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J. M. Burgers

Hydrodynamics Laboratory
California Institute of Technology
Pasadena, California

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J. M. Burgers
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California Institute of Technology, 1950-51

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Foreword to

Report by J. M. Burgers entitled "On Turbulent Fluid Motion"

These lectures were given at the California Institute of Technology during the period January-April 1951.

Since that time several of the subjects touched upon have received further development. This refers in particular to Chapter VII on the application of a mathematical model to illustrate relations characteristic of turbulence. It has been possible to work out a stochastic treatment for this model, applicable for large values of the time. Although certain numerical calculations have not yet been carried through, definite integral expressions can now be given for all quantities appearing in the formulas for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ (see p. 145) of these Lectures), which expressions do do not contain any undetermined coefficients. This opens the way to answer, for instance, questions like those touched upon in Section 67 (pp. 147-148). Also the problems treated in Sections 55 and 56 (pp. 123-128) can now be brought to a conclusion. The following references may be mentioned:

J. M. Burgers, On the coalescence of wave-like solutions of a simple nonlinear partial differential equation; Statistical problems connected with this equation, etc. Proc. Roy. Netherl. Acad. Sciences (Amsterdam) vol. B 57, pp. 45-72, 159-169, 403-433 (1954); and vol. B 58, to appear in the beginning of 1955.

The problem considered in Section 39 (pp. 86-88 of these Lectures), of applying simple linearized equations to the investigation of the turbulent motion accompanying stationary mean flow with shear (with restriction to the case of constant density), has been the subject of a publication "Some considerations on turbulent flow with shear", Proc. Roy. Netherl. Acad. Sciences B 56, pp. 125-147 (1953); this matter, however, has not yet found its final form.

The waviness of the boundary between turbulent and nonturbulent flow, to which was alluded in Section 22 (pp. 54-55), has been the subject of further experimental research by A. A. Townsend, "The eddy viscosity in turbulent shear flow", Phil. Mag. (VII) 41, pp. 890-906 (1950); "The structure of the turbulent boundary layer", Proc. Cambr. Phil. Soc. 47, Pt. 2, pp. 375-395 (1950); and by S. Corrsin and A. L. Kistler, "The free-stream boundaries of turbulent flows," NACA Techn. Note 3133 (Jan. 1954).

The relation between correlations in space and in time has been treated by J. Bass, "Space and time correlations in a turbulent fluid", I and II, Univ. of Calif. Public. in Statistics, vol. 2, pp. 55-84, 85-98 (1954). Experimental work has been done by A. Favre and co-workers, "Quelques mesures de correlation dans le temps et l'espace en soufflerie," Note Technique ONERA (Paris) 22/522 A (1952).

CONTENTS

CHAPTER I

Introduction

1. The following report considers certain aspects of the statistical problems presented by turbulence. Attention has been given in the first place to correlations as defining a structure in the field and to their importance in the phenomenon of diffusion. In attempting to describe the structure of the field by resolving it into simple components, we are faced with the coupling between the components as expressed by the nonlinear form of the hydrodynamic equations of motion. A number of basic effects of nonlinearity in a dissipative system can be studied with the aid of a simplified mathematical model. This model enables us to investigate various problems closely analogous to cases of turbulent fluid motion and presents similar questions referring to spectral resolution. Finally, some aspects of the current statistical theory of isotropic turbulence have been compared with results deduced from the model.

The problem of stability of laminar motion and the resistance problems appearing when turbulent fluid flow is bounded by a wall, have been left aside.

It is difficult to give a clear cut definition of turbulence. In general, we speak of turbulent fluid motion when the precise form of motion of a field does not interest us and we want to know certain average or statistical aspects of the motion only. Such a situation presents itself when the motion is of rather irregular character while, nevertheless, certain general features repeat themselves an indefinite number of times. On the whole, the notion of turbulence also presupposes the presence of some internal mechanism which prevents the irregularities from dying out, at least from dying out rapidly.

An important type is that where the actual motion can be considered as fluctuating about a certain mean state. The mean value of a quantity, taken over a certain period of time, then tends to become independent of the length of the period if it is long enough and the mean value also becomes independent of the instant at which the period is made to begin.

Another type presents itself when the character of the motion appears to be the same in a large part of the field, so that from the statistical point of view the field appears to be homogeneous. We can then make use of average values calculated over a certain domain of the field.

In the case of fluid motion through a cylindrical pipe or in a canal of constant cross section, with constant mean velocity, a combination of the two types is found: at every point of the field the motion can be considered as statistically stationary, while at any instant the general state of the field will be the same over every section of the tube, provided we are far enough away from the entrance and from the exit of the tube. In such cases, mean values can be taken either with respect to time at any given point, or at any given instant of time over a straight line parallel to the axis of the tube. We may also combine the two methods of averaging.

In other cases we find only a stationary pattern with respect to time, but no homogeneity in space. Examples can be found in many types of machinery, for instance when we consider the state of motion in a pump working with constant speed of rotation, constant discharge and constant pressure difference.

There can also be cases where the state of motion is the same over the whole field (so that the field is statistically homogeneous), but where the motion gradually dies out) and consequently is not stationary with respect to time. In these cases space averages must be used.

There are many cases where there is neither stationarity with respect to time, nor homogeneity in space. It may be, however, that the same situation can be reproduced an indefinite number of times by repeating the experiment which brought it about. In such a case we can use averages defined with respect to the series of repetitions of the experiment. Such averages are denoted as "ensemble" averages, since they refer to an ensemble of cases.

2. The number of possible types of turbulent fluid motion is, of course, infinite. Extensive research has been directed mainly to a few groups of types of such nature that the members of a single group appear to be related amongst each other.

Experimentally the most important type of turbulence is derived from the fluid motion found in a long cylindrical tube with constant cross section. The statistical properties of the field are independent of both the instant of time and the coordinate x measured along the tube's axis. For a circular cylinder only, these statistics are symmetric with respect to the axis.

It will not be necessary to enumerate all the types of turbulence related to the one mentioned, since these are well known. In general, one can combine them under the name "boundary layer turbulence".

This form of turbulence is stationary in time, and since there is dissipation of energy as a consequence of viscous friction, the motion must be maintained through the introduction of energy from the outside. In the case of the motion through a tube, energy is derived from the pressure drop in the direction of the axis of the tube. Connected with this circumstance is the fact that the statistical properties are not homogeneous throughout space: although independent of x, they are dependent on the distance of a point from the wall. In the energy relations this dependence appears to be of extreme importance.

Turbulence, homogeneous over a large space, has never been realized in an exact way. It is believed that it can be obtained by suitably stirring a fluid in a large space, for instance in a closed box, and then leaving it to itself, under such conditions that the direct influence of the boundary of the field can be neglected in the interior part of the box. Homogeneous turbulence has been studied extensively from the theoretical point of view because it has simpler properties than nonhomogeneous fields and it is more amenable to mathematical analysis. The theory gives a number of relations that appear to be applicable to those experimental cases which approximate spatial homogeneity.

Again, all forms of homogeneous turbulence are constantly dissipating energy, and since the forms considered here are not supplied with energy from exterior sources (after the initial stirring), the turbulence will die out. The laws governing this dying out form one of the main themes of the theoretical investigation.

One can ask whether it would be possible to obtain fields which simultaneously are homogeneous in space, with respect to all coordinates involved, and stationary in time. Such fields need the action of exterior forces in order to maintain the motion. It will be evident, however, that the spatial pattern of these forces will influence the character of the turbulent motion. Hence homogeneity can be found only when the scale of length used in observation is so large that details of the force distribution are eliminated. Since, in general, one will have to apply forces which are fluctuating or act intermittently, a similar precaution will be necessary with respect to time in order that the field may be considered as stationary. It seems difficult to imagine a proper experimental setup for such a case. One might, perhaps, think of living beings, say a school of fish in the sea, distributed with constant average density and playing about in an irregular way but, on the average, with similar character of motion at every point. The most directly influenced aspect of the turbulence in such a case will be a type of eddies, comparable in extent with the size and the average distance between the fish, and with periods depending on the motion of the fish and on boundary layer phenomena along their bodies. For an observer of dimensions small with respect to the fish, the field will therefore not be homogeneous. But when considered over distances large compared to the mean distance between the fish, the field will present a homogeneous pattern. One can, of course, replace "fish" by any system of forces.

A list of types of turbulence will contain many more cases. As an example of a more complicated type we mention cases where there is a double scale of motions. In the atmosphere, or in the ocean, fields can be found where there are simultaneously present two forms of turbulence, one large scale, the other small scale, with a marked gap in between. The small scale turbulence acts as a kind of eddy viscosity for the large scale motions.

3. It is useful to insert the following observations concerning the idea of statistical homogeneity and the limitations involved in it. For simplicity, consider an infinite one-dimensional series of quantities a_n where n runs from $-\infty$ to $+\infty$ and may represent either equidistant instants of time or equidistant points on a line. The values a_n should follow each other at random, but they should oscillate about a mean value which may be zero.

To make the notion more precise, we imagine that we have not a single series of numbers n, but a large "ensemble" of identical series.

We pick out a particular number, say the nth term, and consider its value for each series of the ensemble. These values can be either positive or negative, large or small; we suppose that there is a distribution law or equivalent probability function which governs the frequency of occurrence of each particular value. For instance, we may take the case where a_n can only have the values -1 and $+1$; it can then be that both values are equally probable. Another case would be where a_n can take all positive and negative values, distributed according to a Gaussian function. Many other cases can be imagined.

We now say that the system is homogeneous when the probability function is independent of n itself, so that it is the same everywhere in a series, whether we consider values of n to the left or to the right of the origin, or at any arbitrary distance from it.

The idea can be generalized. It may be that the probability of a_n having some specified value is dependent on the simultaneous values of a_{n-1} or a_{n+1}, or on the a's at more points in the neighborhood. In such a case there exists a correlation between successive values. We can then again obtain a statistically homogeneous system if these correlations are the same for any value of n. This can be expressed mathematically by saying the correlations are invariant with respect to shifts along the axis of n.

Coming back to the case without correlations (which is the more simple one), the "ensemble mean value" of a_n will be independent of n. This can be seen easily both in the case where only the values -1 and $+1$ occur, and in the case of the Gaussian distribution. In these two cases the mean value will be zero, however, it is also true whenever the probability function is independent of n. We will consequently expect that when, <u>in a single series</u>, we consider the values of a_n for a large number N of consecutive points, the mean value for this group will be zero likewise, with an error which decreases with increasing N and which must approach zero if N is increased without limit. This result will be applicable independently of the starting point of the group of N consecutive points.

The application of probability theory enables us to prove that the ensemble mean value for any group of N consecutive points will be zero. It also enables us to calculate the mean deviation to be expected in any individual group, but it cannot give absolute certainty, for we can imagine that in a group of N consecutive points all values of a_n will be $+1$, and in an infinite set of possibilities such a case necessarily must sometimes occur. The probability of such an event is small and decreases rapidly with increasing N. In the example referring to the Gaussian distribution an exceptionally large value of a_n can occur in a group and may spoil the mean for that group. Of course, cases are also possible where a_n is limited, so that very large values will not occur, or where long runs of values of the same sign are excluded.

Nevertheless, the occurrence of "abnormal" cases will not affect the statistical homogeneity of the system, since it is independent of the situation of the group of N points considered, whether they are to the left or to the right of the origin, or nearby or far distant.

When we turn from the abstract mathematical aspect of the problem to more concrete cases, physical effects can be connected in some way with the values of a_n. Two cases can be distinguished. The physical process may be of such a nature that it tends to smooth out differences; this is the case for instance with diffusion, conduction of heat, etc. The rare occurrence of exceptional sequences will then not greatly affect the physical result and will usually tend towards becoming inobservable in it; but there are also physical processes where the influence of exceptional situations is not smoothed out and where it can be felt very markedly. For instance, let us assume that the values a_n are not quantities at fixed points but represent movable force centers of such nature that centers of the same sign attract each other. The exceptional occurrence of a sequence of all positive a_n will then mean a powerful attractive group which will contract itself and which attracts more and more other positive centers towards it. Hence, it will become very influential in determining the ultimate shape of the system. Its influence can be broken only if somewhere there occurs a larger sequence of positive a_n, which in its turn could be overcome only by a still larger one. In such a case the very rare occurrence of an exceptional sequence can become very effective. Since in an infinite series we must expect that such sequences can turn up anywhere, without preference for a definite value of n, the statistical homogeneity of the system is not lost, although the resulting field may look inhomogeneous over large distances.

It is now of importance to observe that phenomena of turbulence, which to a large extent depend on nonlinear differential equations, have properties which bring them into the second category rather than into the first one. An instructive example is the development of a single series of parallel vortices of equal strength, such as are considered in the investigation of two-dimensional fields of fluid motion. If the vortices are strictly in one line, with exactly constant spacing and with exactly equal strengths, nothing will happen, but the slightest irregularity will become the starting point for a rearrangement, and if somewhere there would be found a group of vortices all having a strength somewhat in excess, or with a spacing somewhat smaller than the normal one, this group will be determinative for the development. Its influence could only be overcome by the effect of a still larger group of abnormal structure, and so on.

4. The type of field which has been used very extensively for investigations concerning approximately homogeneous, decaying turbulence, is the grid-produced wind-tunnel turbulence. Here we are concerned with a type of turbulence produced by exterior forces, the action of which has a stationary character in time. The field obtained is likewise stationary in time, but its distribution in space is not homogeneous. To bring this case into some perspective with respect to the other ones, we describe it as follows: we consider the field of motion in the entrance part of a large tube. It is known that by the use of proper devices the velocity distribution of the incoming current can be made homogeneous and its turbulence can be reduced to about 0.01%, while boundary turbulence which makes its appearance at the walls, does not penetrate to the interior until one is far downstream from the entrance. In the approximately homogeneous and regular entrance current a screen or grid is introduced, with a mesh size small in comparison with the cross section of the current. This screen produces turbulence, and the eddies formed are carried along by the mean motion and gradually die out. When we introduce coordinates, x in the direction of the main motion, y and z parallel to the plane of the screen, the average state of motion at any point will be stationary with respect to time; the motion will depend greatly upon y and z near the screen, but its average pattern will become more and more independent of y and z further downstream. The motion will change gradually with x, but the change may be so gradual that over restricted distances we may consider the motion as being homogeneous with respect to all three coordinates. It is expected

that the statistical character of the turbulence then will also become isotropic.

The type of turbulence obtained in this case is very different from that obtained in a cylindrical tube when the boundary layer turbulence has penetrated so far into the interior that a pattern of motion results which has become independent of x.

Nevertheless, it is believed that certain characteristics of homogeneous, isotropic turbulence will be found in other forms of turbulence. The turbulence found in a tube is one such form, provided one's attention is restricted to the small scale motion. One of the basic problems in turbulence theory is to determine when the statistical properties of homogeneity and isotropy are applicable.

5. In the greater part of the theoretical investigations on turbulence the fluid is considered to be incompressible. The types of motion possible are, consequently, potential motion satisfying the ordinary Laplace equation and incompressible vortex flow.

Cases where changes of volume or density appear are beginning to attract attention. One case is that of a boiling liquid. In the first place, we can imagine a field in which a great number of bubbles are being formed and are disappearing again, in the absence of imposed currents. The appearance and disappearance of bubbles is assumed to occur in a random way; the motions produced will be of the type of those produced by sources and sinks distributed irregularly over the field. One would find here a type of turbulence wholly of "potential" (irrotational) nature, though with a potential not satisfying the ordinary Laplace equation.

A second case is that where currents are imposed upon the field. In such a case the frequency of appearance or disappearance of bubbles usually will be a function of a coordinate, s, measured in the direction of the mean flow. We then come to the type of problem which is of importance in the study of flow with cavitation.

It is probable that, although in these cases we may have fields without rotation, there is still loss of energy through viscosity. Other sources of dissipation would be thermodynamical through some irreversibility in the processes of evaporation and dissipation, through heat conduction, or by means of sound waves, when the compressibility of the liquid is taken into account.

When compressibility of the fluid becomes of great influence, all pressure changes will set up acoustical waves. Part of the energy of the field is in the form of wave motion and part in the form of eddies. When the field is not homogeneous and boundary conditions have to be taken into account, there may be outflow of both types of energy. Such forms of turbulence will be of importance in high velocity boundary flow and also in problems referring to stellar atmospheres or to interstellar gas.

Finally, there are forms of turbulence in which electromagnetic forces play a part and influence the dissipation.

6. It is necessary to give some attention to the problem of how far one should consider turbulence as being "stimulated" by exterior forces or rather as a "spontaneous" phenomenon. In certain cases it looks as if we must consider the turbulence as being stimulated. This was the case when we considered the fish; also the production of turbulence in a wind tunnel by means of a grid can be brought under this heading. We can then assume that the irregular and fluctuating character of the turbulent motions is primarily a consequence of the randomness of the forces. A different case is presented by turbulence in the flow through a tube and generally with boundary layer turbulence. The exterior force driving the flow, in the case of motion through a tube, i.e., the pressure gradient, can produce a completely regular motion, the so-called laminar or Poiseuille flow. The fact that actually irregular flow is obtained (provided the Reynolds number for the case under consideration exceeds a certain critical value), is a consequence of an inherent instability of the laminar flow at high Reynolds numbers. Slight deviations from the mathematically exact pattern can then lead to a complete change of the whole field. In such a case one can speak of "spontaneous" appearance of turbulence.

When we attempt to look more closely into the two cases, we observe that in both it is useful to describe the actual state of the field at any moment as a superposition of elementary types of motion, as can be done, for instance, by means of Fourier series or Fourier integrals. (Any other system of normalized solutions of a linear differential equation could serve the same purpose.) It is always possible, at any given instant, to obtain such a resulution of the field. When the analysis is repeated at another instant, we obtain other values for the amplitudes and we can describe the history of the field by means of the time dependence of the various amplitudes.

In those mechanical systems, the behavior of which is governed by linear equations, a form of resolution can be found in which all the components are completely independent of each other. The time dependence of each amplitude is then determined by an equation in which the other amplitudes do not enter; each amplitude consequently follows its own course. This course will depend on the way in which the corresponding component has been stimulated or is maintained by an influence deriving from exterior forces; it may grow or decay, and it may be stimulated repeatedly, but it is always independent of its companions. We cannot properly speak of turbulence in such a system; if the aggregate effect of all components looks rather turbulent, this turbulence is fully dependent on the way in which the system has been stimulated.

In the case of systems governed by nonlinear equations, such a resulution into independent components is impossible. Every method of resolution gives a series of components whose equations of motion are interrelated in such a way that they cannot be separated. Hence, every component is coupled with all others. If at some initial instant only one, or a few components are excited, other components will appear soon afterwards. In general, we can expect that always the whole spectrum will arise. This is the case for those hydrodynamical systems where turbulence has been observed.

The problem of making a distinction between "stimulated" and "spontaneous" turbulence now can be formulated in a different way: we can ask whether the resulting turbulent motion largely reflects features present in the system of stimulating forces, or whether it is nearly independent of it.

7. Since the coupling between the various components of a turbulent motion depends on terms of the second degree in the amplitudes, it is very weak so long as all amplitudes are very small. If we neglect the coupling for very small amplitudes, we obtain linear equations in which the components are independent of each other. These equations enable us to find the

development of each separate component, once it has been stimulated. Now a difference can be observed between two classes of systems: in one class every component, after having been stimulated, goes through a certain history which ultimately ends with decay; in the other class it is found that the components can pick up energy from exterior sources and that with a certain number of components this takes place to such a degree that these components will increase exponentially. It is the latter case which presents itself with pipe flow. Here the components of turbulence can take up energy from the mean motion (which in itself is maintained by the pressure gradient), and for a certain number of components (depending on the magnitude of the Reynolds number) the linear equations predict exponential increase. We shall illustrate this by means of an application of a simplified mathematical model, to be considered in sections 73 which follow.

It will be understood that the latter phenomenon is of extreme importance. Since in any case of laminar flow it is impossible to eliminate all disturbing effects, there is always a chance that some of these exponentially increasing amplitudes will be stimulated provided, of course, that the Reynolds number exceeds its critical value. Once this has occurred, the amplitudes of these components will grow and soon a state will be reached where the coupling terms can no longer be neglected, and from then onward all - or at least a great number - of other components will come into play. The state of turbulence which comes about in this way will be largely or even completely independent of the form and the magnitude of the original stimulation. Hence in such a case the term "spontaneous" turbulence can be considered as adequate.

On the other hand, in those cases where components cannot derive energy from outward sources and where they ultimately will always decay, such an independence of the stimulating force system usually will not be arrived at. It is possible that the amplitudes of certain components, once they have been stimulated, first will increase to such a degree that the coupling terms come into play, and that a great number of other components will take part in the motion. The resulting motion thus is really turbulent, but if there is no further stimulation, decay will set in and the final state may be one of rest or of laminar motion. Such a case presents itself, for instance, when a fluid in a closed space is stirred and then is left to itself. Repeated or continuous stirring is necessary to produce some stationary form of turbulence. In these cases it is probable that features of the stirring forces will have a determining influence on the character of the resulting turbulent motion, and the term "stimulated" turbulence for such cases is appropriate.

8. To follow the full history of each component, in a system where the various components are coupled by means of nonlinear terms, is not possible with present mathematical methods; but it has already been mentioned in the beginning that we are not interested in a precise description of any field; what we need are certain averages, statistical properties, etc. Hence it follows that there is need for a statistical theory of turbulence.

Statistical problems in mechanics have been studied widely in connection with the molecular theory of matter and in radiation theory. All these investigations, however, referred to systems which are conservative, so that the energy once given to them is retained. The statistical problem then can be interpreted as a problem of the average distribution of the total energy over the various degrees of freedom of the system. It is well known that powerful and elegant methods have been developed to solve these problems.

The situation is different with turbulence because of the essential part played by dissipation. All hydrodynamic systems dissipate energy, and in particular in the case of pipe flow, there is a balance between the energy introduced into the flow by the pressure gradient and the dissipation of energy through viscosity. It does not help us to include heat motion in order to explain the loss of mechanical energy by a gain of energy of molecular motions, since in that case we still must consider the removal of heat out of the system. Inflow and outflow of energy are essential features. The methods of classical statistical mechanics, as developed for conservative systems, do not apply to the nonconservative, dissipative systems which are presented by hydrodynamic turbulence. Many investigators are attempting to find new methods of approach. Much success has been obtained in a number of detached problems, but as yet there is no basic theory embracing all aspects in a comprehensive way.

It is true that a number of practical questions referring to hydrodynamic resistance, Reynolds stresses in shear flow, diffusion, mixing, heat transfer, are dependent mainly on those components of the turbulent field which present a rather coarse pattern and which are only slightly affected by viscous decay. It is impossible, however, to treat these components separately from the rest. It appears that energy is detracted from these components by the whole aggregate of components of finer pattern, and the whole spectrum is involved in formulating the balance of energy for any single component. Since the behavior of the components of finer pattern is very much influenced by viscous decay, it is impossible to leave viscosity out of the picture. This forms one of the great difficulties of the statistical theory of turbulence.

Summing up, we may say that the problems before the investigator lead to two important groups of questions:

(a) To find the way in which practical quantities such as the transfer of momentum, suspended material or heat depend on the character of the spectrum.

(b) To define the spectrum more precisely and to find the relations which govern the distribution of energy over it.

Serious difficulties present themselves in both groups of problems.

CHAPTER II

Correlation Functions

9. To obtain insight into the nature of certain characteristics of the turbulent field, we consider a problem referring to the diffusion of particles. We assume that the particles have the same density as the fluid and that they are sufficiently small in order to follow the motion of the elements of volume of the fluid without time lag. We suppose that the motion of a large number of particles is observed, all starting from the same point of the field. If there is a mean motion in the field, the particles will be carried along by this mean motion. To simplify the problem as much as possible we suppose that the mean motion is stationary, rectilinear (and parallel to the x-axis) and uniform over a certain domain so that it represents a translation in a definite direction with constant velocity. By introducing a coordinate system moving with the mean flow, we can eliminate its effect so that we are only concerned with the turbulent motion superposed on it.

The turbulent motion can be in three coordinates. We restrict our attention to only one of these coordinates, say y.

Observation of the motions of the individual particles will give data which must be reduced by statistical evaluation. We consider the positions of the particles after a certain duration T since they have started from their origin. For any value of T, the same for all particles, we can represent the values of the coordinates y of the particles in a diagram. The distribution obtained may appear to be simply Gaussian. If this is the case, the shape of the curve can be characterized by a single parameter, for which one usually takes the average value of y^2. If the distribution differs from a Gaussian one, $\overline{y^2}$ still is an important parameter.

We will investigate how this mean value, which is a function of T, is related to properties of the field.

10. We consider the velocity of a single particle in the y-direction as a function of two variables: the time at which the particles started and the duration T elapsed since that moment. Now:

$$y = \int_0^T dT' \ v(t_o, T') \ ;$$

hence we have:

$$y^2 = \int_0^T dT' \int_0^T dT'' \ v(t_o, T') \ v(t_o, T'') \ .$$

The double integral combines all values of T' from 0 to T with all values of T'' from 0 to T. It can be pictured as the integral over a square of side T in a T', T''-plane. Before carrying out the integration, we calculate the mean value of $v(t_o, T') \ v(t_o, T'')$ for a large number of particles and write:

$$\overline{y^2} = \int_0^T dT' \int_0^T dT'' \ \overline{v(t_o, T') \ v(t_o, T'')} \ .$$

We now apply the following reasoning. We have assumed that the field is stationary and homogeneous. The particles wandering over the field during a large part of their course will be subjected to random influences of the same statistical nature. This has the consequence that the mean value $\overline{v(t_o, T') \ v(t_o, T'')}$ will be independent of t_o and that it can depend only on the <u>time difference</u> $\tau = T' - T''$. More precisely it will depend only on the absolute value of this difference, $|T' - T''|$, since the order in which the two positions are taken is immaterial. If the turbulence, although being stationary in time, would not be homogeneous and if the particles would wander too far away into regions where their behavior would be different, the values of T' and T'' themselves would enter. We will limit ourselves, however, to the case where τ is the only relevant variable.

We write:

$$(1) \qquad \overline{v(t_o, T') \ v(t_o, T'')} = R_v(\tau) \ .$$

For $\tau = 0$, which means $T' = T''$, we have:

$$R_v(0) = \overline{v^2} \ .$$

It can be proved that the value of $R_v(\tau)$ never can be greater than $R_v(0)$. This is shown as follows: we write $v(t_o, T') = v_1$; $v(t_o, T'') = v_2$. Now:

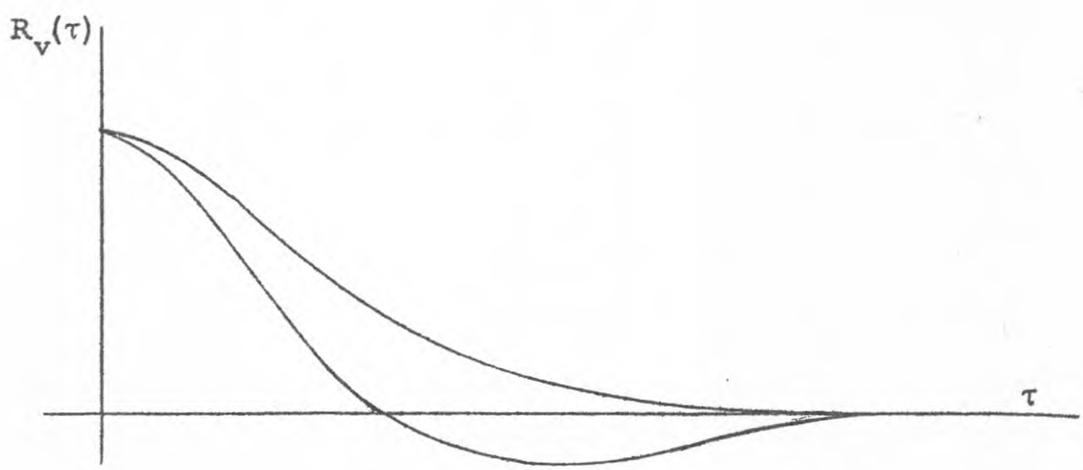
$$(v_1 - v_2)^2 = v_1^2 + v_2^2 - 2\, v_1 v_2 \; .$$

Taking mean values, and noticing that the mean value of v_1^2 is the same as that of v_2^2 and that in fact both are equal to $R_v(0)$, we find:

$$\overline{(v_1 - v_2)^2} = 2 \left[R_v(0) - R_v(\tau) \right] \; .$$

This proves the theorem, since the left-hand side of the equation cannot be negative.

It is possible that $R_v(\tau)$ assumes negative values for a certain domain of values of τ. This depends on the statistical laws of the turbulent field, which can be different for various cases. The diagram below gives some possible forms of $R_v(\tau)$-curves, or, as they are called, correlation curves. In general, we must expect that $R_v(\tau)$ goes to zero when τ becomes large enough. (An exception is obtained in the case of a purely periodic motion, but this will not be found in turbulence.)



One can now calculate the mean value $\overline{y^2}$ from the integral:

$$\overline{y^2} = \int_0^T dT' \int_0^T dT'' \, R_v(\tau) \; .$$

As a consequence of the symmetry of $R_v(\tau)$ this can be written:

$$\overline{y^2} = 2 \int_0^T dT' \int_0^{T'} dT'' \, R_v(\tau) \; .$$

By partial integration it can be further transformed into:

$$(2) \quad \overline{y^2} = 2 \int_0^T dT' \int_0^{T'} d\tau \, R_v(\tau) =$$

$$= 2 \left\{ T \int_0^T d\tau \, R_v(\tau) - \int_0^T d\tau \cdot \tau \cdot R_v(\tau) \right\} \; .$$

We can expect that the mean value $\overline{v(t_o, T') \; v(t_o, T'')}$ goes to zero when the time difference τ increases without limit. In physical problems it is supposed that the function $R_v(\tau)$ goes to zero sufficiently rapidly to make both integrals convergent when $T \longrightarrow \infty$. This means that both integrals will approach a constant value when T becomes large. For these values we shall write:

$$(3) \quad \int_0^\infty d\tau \, R_v(\tau) = D \; ; \qquad \int_0^\infty d\tau \cdot \tau \, R_v(\tau) = DT_o \; .$$

In this way we obtain:

$$(4a) \quad \overline{y^2} = 2D\,(T - T_o) \qquad \text{for large } T.$$

Expressions valid for very small values of T can be obtained if we may assume (which is usually true in physical problems) that $R_v(\tau)$ remains very near to $\overline{v^2}$ so long as τ is small. We can then consider it as a constant in carrying out the integrations and obtain:

$$(4b) \quad \overline{y^2} = \overline{v^2} \, T^2 \qquad \text{for small } T.$$

This can also be written:

$$\overline{y^2} = \overline{(vT)^2} .$$

It expresses the fact that when T is so small that practically every particle retains a constant velocity, the mean square value of the distance will be proportional to the square of T. For long intervals, on the other hand, the mean square value of the distance becomes a linear function of T, as shown by equation 4a.

We shall call the function $R_v(\tau)$ the correlation function for the y-movement of a particle. In many cases the correlation function is defined as the quotient $R_v(\tau)/\overline{v^2}$; for $\tau = 0$ this quotient has the value unity. The function defined in this way is denoted as a normalized correlation function.

Similar functions can be defined for the movement in the direction of z or of x (after the mean motion has been eliminated).

The particular correlation functions (whether normalized or not) considered here, refer to the history of a single particle. According to the supposition introduced initially, they also refer to the history of a single element of volume of the fluid. Looking at them from the latter point of view, we say that they represent the Lagrangian correlation, because it is the Lagrangian description of a field of fluid motion that gives attention to the history of the individual elements of volume.

11. Eulerian correlations. Most of our information about the state of motion of a fluid, both that resulting from experimental investigation and that obtained from theoretical deduction, is given in the Eulerian description where velocities are recorded as a function of coordinates fixed in space and of the time. This description is not concerned with the history of a single element of volume.

Starting from the Eulerian description, a different system of correlation functions can be constructed. These new "Eulerian" correlation functions refer to relations in space or to relations in time (or to both) and they are characteristic for the field.

Let us first take the case of a homogeneous field of turbulence. We can then consider the product $v_1 v_2$, where v_1 is the velocity (or a velocity component) at a particular point of space, P_1, and v_2 the velocity (or the same component) at another point P_2, both for the same instant of time, t.[1] In particular we may consider the y-component of the velocity, and the two points P_1 and P_2 may have the same x- and z-coordinates but different y-coordinates. Let us denote the latter by y and $y + \eta$, respectively. We now consider various pairs of points P_1, P_2, always with the same difference η between the y-values, and take the mean value of the product $v_1 v_2$. The quantity obtained in this way will be denoted by:

$$(5) \qquad \overline{v_1(y)\ v_2(y+\eta)} \quad = \quad S_1(\eta) \ .$$

In general this function can depend on t, the instant for which the correlation is calculated.

If the field is stationary in time, we may also take the mean value over a certain interval of time and obtain the same result.

In a field which is stationary in time, but not necessarily homogeneous in space, we can likewise calculate mean values $\overline{v_1 v_2}$, where v_1 and v_2 are values of the velocity at the <u>same</u> point P of the field, but at two different instants separated by a constant interval τ. The mean value then obtained will be denoted by

$$(6) \qquad \overline{v_1(t)\ v_2(t+\tau)} \quad = \quad S_2(\tau) \ .$$

It will be seen that the function considered here is different than the Lagrangian correlation because v_1 and v_2 in the present case are not for a single element of volume, followed in its course, but are velocities measured at a fixed point of the field and consequently are for different elements of volume. If the field is not homogeneous, the mean value S_2 will depend on the position of the point P and thus be a function of the coordinates of this point.

If the field is both homogeneous in space and stationary with respect to time, it is possible to define a more general type of correlation function

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[1] In the complete theory of correlations in a field of flow, correlations between different components are also considered. The two components, for instance u and v, can either be measured simultaneously at a single point or at different points.

by making v_1 refer to a point P_1 with coordinates x, y, z at an instant t, and v_2 to a point P_2 with coordinates $x + \xi$, $y + \eta$, $z + \zeta$ at an instant $t + \tau$. We now again take various pairs of points, keeping ξ, η, ζ, τ all constant, but shifting x, y, z and t. It is not necessary to shift all the latter quantities; under the assumptions of statistical homogeneity and stationarity, it will be sufficient to determine a time mean value (keeping x, y, z constant) or to take a mean value with respect to x, etc. The following notation can be used for a mean value of this general type:

$$\overline{v_1(x, y, z, t)\, v_2(x + \xi,\ y + \eta,\ z + \zeta,\ t + \tau)} = S(\xi, \eta, \zeta, \tau). \tag{7}$$

When the assumptions of homogeneity and stationarity do not apply, mean values and correlations can only be defined with the aid of ensemble averages. We must then assume that instead of a single field of flow, a great many similar fields are given, in all of which we have the same system of coordinates x, y, z, while all of these fields display their history on the same time scale. We can now consider a set of values of x, y, z, t and a second set $x + \xi$, $y + \eta$, $z + \zeta$, $t + \tau$, and ask for the values of v_1 and v_2 at the corresponding points of all these fields. For each field we calculate $v_1 v_2$ and then take the average over all fields.

The result obtained in general will depend on all eight coordinates, so that it will be a difficult object to handle from a mathematical point of view. If the same procedure of ensemble averaging is applied to a homogeneous field, the coordinates x, y, z will drop out; if the field is stationary, the time t will drop out. In that case we come back to the correlation functions considered before.

The consideration of ensembles lead to the introduction of probability functions for the velocity at a given point-instant x, y, z, t of a field. In the Appendix to this chapter we will consider certain general problems which turn up when correlations are studied by means of this method.

12. Scales defined by correlation functions. When we consider the curve representing the behavior of a correlation function, two characteristics are of great importance. One refers to the distance at which the correlation practically vanishes; the other refers to the behavior of the

correlation function for very small values of the distance.

As a first example, we take the Eulerian space correlation function $S_1(\eta)$. If S_1 becomes exactly zero for values of η exceeding some threshold η_m, we can use η_m as a measure of the maximum distance up to which correlation may be perceptible. If S_1 decreases gradually, so that a definite limit cannot be given, an average measure can be defined by means of the integral:

$$L = \frac{1}{\overline{v^2}} \int_0^\infty S_1(\eta)\, d\eta , \tag{8}$$

which represents the area of the correlation curve divided by its maximum ordinate. (If instead of S_1 the corresponding normalized correlation function is used, with maximum ordinate equal to unity, L becomes the area of the curve.) The quantity L does not give the maximum distance to which correlation extends, but it fixes its order of magnitude and it has the advantage that the definition is precise. There are, however, cases in which the function S_1 changes sign and where the integral has the value zero. Formula (8) cannot be applied in such a case. A convenient measure may then be found by taking the value of η for which S_1 becomes zero for the first time. It is usual to consider L (or one of the other quantities mentioned) as indicating the "macroscale" of turbulence. It gives a measure for the average size of domains as coherent motion or eddies. In certain fields of flow the scales for the different directions may be different.

To obtain the second quantity, we consider the relations:

$$\frac{\partial S_1}{\partial \eta} = \overline{v(y) \left(\frac{\partial v}{\partial y} \right)_{y+\eta}}$$

$$\frac{\partial^2 S_1}{\partial \eta^2} = \overline{v(y) \left(\frac{\partial^2 v}{\partial y^2} \right)_{y+\eta}}$$

If we take $\eta = 0$, they reduce to:

$$\left(\frac{\partial S_1}{\partial \eta}\right)_o = \overline{v \frac{\partial v}{\partial y}} = \overline{\frac{\partial}{\partial y}\left(\frac{v^2}{2}\right)}$$

$$\left(\frac{\partial^2 S_1}{\partial \eta^2}\right)_o = \overline{v \frac{\partial^2 v}{\partial y^2}}$$

The first mean value is zero, which proves that the correlation curve must have a horizontal tangent at the origin. With reference to the second formula, we observe that:

$$v \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(v \frac{\partial v}{\partial y}\right) - \left(\frac{\partial v}{\partial y}\right)^2 .$$

Since in a homogeneous field the mean value of a derivative with respect to y is zero, we find:

$$\overline{v \frac{\partial^2 v}{\partial y^2}} = -\overline{\left(\frac{\partial v}{\partial y}\right)^2} .$$

Hence the expression for $(\partial^2 S_1/\partial \eta^2)_o$ can be written:

$$\left(\frac{\partial^2 S_1}{\partial \eta^2}\right)_o = -\overline{\left(\frac{\partial v}{\partial y}\right)^2} . \tag{9}$$

It follows that the development of $S_1(\eta)$ in the neighborhood of $\eta = 0$ has the form:

$$S_1(\eta) = \overline{v^2} - \frac{1}{2}\eta^2 \overline{\left(\frac{\partial v}{\partial y}\right)^2} + \cdots \tag{10}$$

$$S_1(\eta) = \overline{v^2}\left(1 - \frac{\eta^2}{2\lambda^2} + \cdots\right) \tag{11}$$

$$\lambda^2 = \overline{v^2} \Big/ \overline{\left(\frac{\partial v}{\partial y}\right)^2} \tag{12}$$

The quantity λ obtained from this equation is a measure for the distance over which S_1 begins to deviate markedly from its maximum value. It becomes smaller when the mean value of the square of the derivative $\partial v/\partial y$ increases. Usually λ is considered as determining the "microscale" of a turbulent field. This microscale is of particular importance in formulas expressing the dissipation of energy in a turbulent field, since the dissipation depends on the squares of the velocity gradients with which also λ is connected. In many cases of turbulent motion it is found that the orders of magnitude of the two scales are widely different, to a degree depending on the Reynolds number for the field of flow.

Similar scales can be defined with reference to the other correlation functions. If we consider the Eulerian time correlation $S_2(\tau)$ we can deduce a "macroscale" in time, connected with the maximum time interval during which the velocity at a given point approximately retains its value; and a "microscale" in time, connected with the rate of variation $\partial v/\partial t$.

We can also deduce time scales from the Lagrangian correlation function. The scales then refer to the history of an element of volume which is followed in its motion. Both scales will be connected in some way with the inertia of an element of volume which, popularly speaking "makes it reluctant to change its velocity".

13. A problem of great importance in turbulence is whether it is possible to find relations between the Lagrangian and the Eulerian correlation functions. The problem is extremely difficult and has not been solved. The difficulty is twofold: on the one hand there is the general problem of relations between Lagrangian and Eulerian mean values; on the other hand, in order to pass from the Eulerian description of a field of flow to the Lagrangian description of the motion of an element of volume, it is necessary to integrate a differential equation and to discuss the behavior of its solutions "in the large", that is, for arbitrarily increasing values of the time.

A relatively simple relation between Eulerian and Lagrangian mean values of the same physical quantity, for instance a velocity component, can only be expected when the statistical character of the field is both homogeneous and stationary. Otherwise the Lagrangian paths of elements of volume will wander through domains with different statistical properties and no simple conclusion can be obtained; but even when we assume homogeneity and stationarity a problem remains.

When the field is homogeneous and stationary, the Eulerian mean value of a particular velocity component, defined either as a mean with respect to space or as a mean defined with respect to time, will be always and everywhere the same. It is often assumed that in such a case also the Lagrangian mean value of this velocity component would have the same value. Simple cases, however, do not (or not always) confirm this assertion. We mention the following example:

(a) $$v = A \cos(\omega t + \lambda y), \quad \text{with } A < \omega/\lambda .$$

When this expression is considered as defining a velocity field in the Eulerian description, it is evident that the Eulerian mean value of v will be zero, the same whether it is taken with respect to time for fixed y, or with respect to y for fixed t. Now, to pass from the Eulerian description to the Lagrangian one it is necessary to integrate a differential equation which in the present case has the form:

$$dy/dt = v(y,t) = A \cos(\omega t + \lambda y).$$

(In the more general case of flow in three dimensions, a system of simultaneous differential equations must be solved.) The example has been chosen so that the integration can be performed without difficulty; we take $z = \omega t + \lambda y$ as a new variable, for which we obtain the differential equation:

$$dz/dt = \omega + \lambda A \cos z .$$

The latter equation can be integrated when t is considered to be the unknown; the result can be put into various forms, for instance:

$$\sqrt{\frac{\omega - \lambda A}{\omega + \lambda A}} \operatorname{tg} \frac{z}{2} = \operatorname{tg}\left(\frac{t-s}{2} \sqrt{\omega^2 - \lambda^2 A^2}\right) .$$

Here s is the integration constant which has a fixed value for any path of an element of volume. For constant s, let t increase with $2\pi(\omega^2 - \lambda^2 A^2)^{-1/2}$. Then the tangent function on the right-hand side will go through one period;

hence the same must be true for $tg(z/2)$ and z consequently will increase by the amount 2π. Knowing the increase of z and that of t, we can calculate the increase of y; we find that y changes by the amount:

$$\frac{2\pi}{\lambda}\left(1 - \frac{\omega}{\sqrt{\omega^2 - \lambda^2 A^2}}\right).$$

Thus y does not return to its original value ; there is a "secular change" (in the present case: a secular decrease) of y, and it follows that the mean velocity along the path of an element of volume - that is the Lagrangian mean velocity - is not zero, but is negative. If we assume A to be a small quantity, the decrease of y per period will be approximately $\pi \lambda A^2/\omega^2$ and since the duration of the period is approximately $2\pi/\omega$, the mean velocity becomes:

$$v_{\text{Lagrange}} = -\frac{1}{2}\lambda A^2/\omega .$$

It is possible that the result obtained here is due to the rather simple and regular character of the Eulerian expression which we have assumed for the velocity. This function did not have any random element in it. However, when it is attempted to generalize the reasoning by considering an expression of the type:

$$v = \Sigma A_n \cos(\omega_n t + \lambda_n y) ,$$

where the various ω_n might be incommensurable, integration of the differential equation in finite terms becomes impossible.

Another simple example which may be considered is:

(b) $$v = A \cos \omega t/(1 + a \cos \lambda y) , \text{ with } a < 1.$$

The differential equation which must be integrated in order to obtain the path of an element of volume, in this case can be written:

$$(1 + a \cos \lambda y) \cdot dy/dt = A \cos \omega t.$$

The integral of this equation has the form:

$$y + \frac{a}{\lambda} \sin \lambda y = s + \frac{A}{\omega} \sin \omega t ,$$

where again s denotes the integration constant, which has a fixed value for every path. If α is a small quantity, it is possible to develop y into a Fourier series, proceeding according to sine-functions of multiples of λy. Even without doing this it will be seen that in the present case y is a purely periodic function of the time (for constant s), so that there is no "secular" increase or decrease of y when we follow a path. Hence the Lagrangian mean value of the velocity in this case is truly zero. But when we return to the original expression (b) and ask for the Eulerian mean value of the velocity, we cannot say that the mean value with respect to y, for a fixed value of t, is zero. (The Eulerian time mean value for fixed y is zero.) Hence again it appears that the assumption stated in the second paragraph of this section cannot be substantiated with the aid of simple examples.

14. It will be evident that with such difficulties already appearing in connection with simple mean values, the problem of the relation between Eulerian and Lagrangian correlations becomes even more difficult. When arbitrary general expressions are used for the velocity, there is no possibility of integrating the equation in finite terms, and series developments, in consequence of their limited domain of convergence, do not give a satisfactory basis for a discussion of the properties of the integrals "in the large".

Nevertheless, the discussion of certain properties of the field of motion may require some form of relation. In Chapter V we shall develop a type of calculation which, for a particular kind of fields, will allow us to derive some Eulerian mean values with a reasonable degree of certainty from data referring to the Lagrangian velocity correlation. In the fields to be considered the elements of volume carry some specific property; for instance the concentration of suspended or dissolved material, temperature, momentum, etc. The elements take the property along with them, but at the same time there is a certain amount of exchange of this property between neighboring elements. It is supposed that over a certain domain the field can be treated as homogeneous and stationary. When we look at a particular point of the field or at a fixed small region, it has, therefore, a sense to speak of the mean value of the concentration, or of the temperature, etc., at that point or in that region. In the case of importance this mean value is not constant throughout the whole domain, but has a certain gradient, say in the direction of z. The gradient is treated as a constant over the domain.

In this domain now, a fixed unit plane area normal to the z-axis is considered, and it is required to find the average transport of the specific property through this unit area in unit time. For this purpose it is necessary to find the Eulerian mean value:

$$\overline{w\,c}$$

at the points of the plane area, where w is the velocity component in the z-direction of an element of volume at the instant it crosses the plane area and c is the concentration of the property in the element at the same instant. The mean value must be taken over all elements of volume crossing the plane area, either simultaneously or during a certain period of suitably chosen length.

The theory to be developed in Chapter V will show how the value of the concentration c at the instant t when the element of volume crosses the unit area, can be calculated from the equation governing the exchange process during its history previous to t. The result depends on the velocity of the element during its history previous to t and on the gradient $d\bar{c}/dz$ of the mean concentration $\bar{c}$. The following expression is obtained:

$$c = \bar{c} - \frac{d\bar{c}}{dz} \int_0^\infty dt' \; e^{-\lambda t'} \; w(t-t') \; .$$

Here λ is a coefficient determining the rate of exchange of the property under consideration (concentration of suspended material, temperature, momentum, etc.) between neighboring elements of volume. Since in the case treated the mean value of w itself at the fixed plane area is assumed to be zero, it is found that the mean value of $w\,c$ is given by:

$$\overline{w\,c} = - \frac{d\bar{c}}{dz} \int_0^\infty dt' \; e^{-\lambda t'} \; \overline{w(t-t')\,w(t)} \; .$$

Here under the integral sign occurs the mean value $\overline{w(t-t')\,w(t)}$, which by its definition is a Eulerian mean value, since it must be determined at a definite location of the field (the unit plane area), for all elements of volume which cross this plane area during a certain length of time. The two factors of the product, however, refer to the velocities of a single element of volume and thus refer to its history in the Lagrangian sense. It is evident, therefore, that the mean value will depend on the Lagrangian velocity correlation for an element of volume.

The customary assumption now is that, when the field can be treated as homogeneous over a certain domain, it is not necessary to keep strictly to the instants at which the elements of volume cross the unit plane area. If we take into consideration elements shortly before or shortly after the instant of crossing, the degree of correlation will not materially differ. We may say, therefore, that what we need is the Lagrangian correlation $\overline{w(t - t')\, w(t)}$ for the elements of volume in the neighborhood of the unit plane area, without further qualification. We shall make use of this reasoning in Section 31.

Additional Remark. We will come back to one difficulty concerning the relations between Lagrangian and Eulerian mean values. In certain cases it might be desirable to have a relation between the micro-timescales, to be deduced from the Lagrangian correlation and from the Eulerian time correlation. According to the equations of Section 12, this would require a relation between the mean values of $(dv/dt)^2$ and $(\partial v/\partial t)^2$. Now we have the well known relation:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y},$$

where dv/dt is the time derivative along the path of an element of volume in the Lagrangian sense, and $\partial v/\partial t$ is the local time derivative in the Eulerian description. Squaring on both sides:

$$\left(\frac{dv}{dt}\right)^2 = \left(\frac{\partial v}{\partial t}\right)^2 + 2v\frac{\partial v}{\partial y}\frac{\partial v}{\partial t} + v^2\left(\frac{\partial v}{\partial y}\right)^2 .$$

It looks as if this equation can be applied to obtain a relation between $\overline{(dv/dt)^2}$ and $\overline{(\partial v/\partial t)^2}$. But when it is noted that the first mean value must be determined along a path and the second one at a given point, it will be evident that a direct calculation of both quantities by means of a single process cannot be carried out.

CHAPTER II

Appendix

Concerning Some Applications of Probability Functions

In certain cases it is convenient to consider a field of motion as an instance of an ensemble of similar fields. All fields of the ensemble will be of the same geometrical character and are subjected to the same system of forces; they have identical mean values for the velocity of flow, etc., and are identical in their statistical behavior, but they differ in individual values. This means that if we consider the value of v for fixed values of the coordinates and the time (say, for fixed values of y and t) in all these fields, v will vary in a random way from one field to the next. The values of v, however, cannot be completely arbitrary since the behavior of the field is governed by physical laws; hence they must be subjected to some probability law.

It is impossible here to give an adequate treatment of the application of probability functions to ensembles; we will restrict our attention to a few examples in the hope that they may illustrate some of the main ideas.

To take a simple case first: It may be that the probability that in any one of the fields of the ensemble, for given y and t, the velocity v will have a value, say between v and $v + dv$, given by a Gaussian function:

$$\sqrt{\frac{a}{\pi}}\, e^{-a(v - v_o)^2}\, dv.$$

Here a and v_o are constants, the latter one giving the mean value of v, the former one connected with its spreading. Both a and v_o can be functions of y and t, depending on the physical laws governing the field. If we should know the form of these functions, we could find the mean value of v and the amplitude of its fluctuations around the mean value for all points y and all instants t. In many cases this will form a satisfactory statistical description of the field. The mean square value of the deviation of v from its mean value v_o is found to be:

$$\overline{(v - v_o)^2} = 1/2a \ .$$

In the particular case where the field is stationary and homogeneous, a and v_o would be independent of y and t. If the mean value of v is zero, v_o must be zero.

In the description by means of a single probability function for each point y, t something nevertheless is lacking. It does not give information about the possibility of correlations between simultaneous values of v at different points of the same field, or between successive values of v at a single point of a field. If probability function of the type given above would embody all our knowledge, we would be led to assume:

$$S(\eta, \tau) = 0$$

for η or τ, or both, differing from zero.

From the physical point of view it is absurd to imagine that the velocity could change from point to point or from instant to instant without any restriction, even when the points or the instants are very close together. Hence in order to obtain a satisfactory description in terms of probability functions, more complicated expressions are necessary.

In many cases we can take account of correlations by using a Gaussian probability function bearing on the simultaneous occurrence of particular values of v at two points P_1 and P_2. If for simplicity we assume that the mean value of v is zero everywhere, we can use an exponential function of the type:

$$\overline{\omega} = e^{-av_1^2 + 2b\, v_1 v_2 - c\, v_2^2}$$

The constant factor before the exponential function has been omitted in order to simplify writing. If desired, it can be restored by observing that the integral of the correctly written function, extended over all possible values of v_1 and v_2, that is over all values of v_1 and v_2 from $-\infty$ to $+\infty$, must be equal to unity. The constants a, b, c in the exponent in general will be functions of the coordinates y_1, t_1, y_2, t_2 of the points P_1 and P_2. When the field is statistically homogeneous and stationary, they will reduce to functions of the intervals $\eta = y_2 - y_1$; $\tau = t_2 - t_1$.

The mean value of v_1^2 is obtained from the equation:

$$\overline{v_1^2} = \frac{\iint v_1^2 \omega \, dv_1 \, dv_2}{\iint \omega \, dv_1 \, dv_2} \, .$$

The integration in both double integrals is extended from $-\infty$ to $+\infty$ for both v_1 and v_2. Similar expressions can be used to find $\overline{v_2^2}$ and $\overline{v_1 v_2}$. The calculation can be made in a relatively simple way, if we introduce the quantity:

$$I = \iint_{-\infty}^{+\infty} \omega \, dv_1 \, dv_2 \, .$$

This integral can be found most easily, if the exponent of the e-function is written as a sum of squares; we then have:

$$I = \iint_{-\infty}^{+\infty} e^{-a(v_1 - b\, v_2/a)^2 - (c - b^2/a)\, v_2^2} = \frac{\pi}{\sqrt{ac - b^2}}$$

Now from the original expression for ω it is not difficult to deduce that:

$$\iint_{-\infty}^{+\infty} v_1^2 \omega \, dv_1 \, dv_2 = -\frac{\partial I}{\partial a} \, .$$

It follows that the mean value $\overline{v_1^2}$ is given by:

$$\overline{v_1^2} = -\frac{1}{I} \frac{\partial I}{\partial a} \, .$$

Similar formulas apply to the other two mean values.

It is possible to calculate the mean values of v_1^2, v_2^2 and $v_1 v_2$ from the probability function $\overline{\omega}$. The following results are obtained:

$$\overline{v_1^2} = \frac{c}{2(ac - b^2)} \; ; \quad \overline{v_2^2} = \frac{a}{2(ac - b^2)} \; ; \quad \overline{v_1 v_2} = \frac{b}{2(ac - b^2)} \; .$$

If $a = c$, we find $\overline{v_1^2} = \overline{v_2^2}$. Whether this is the case or not, we can define the normalized correlation between v_1 and v_2 by means of:

$$\frac{\overline{v_1 v_2}}{(\overline{v_1^2} \cdot \overline{v_2^2})^{1/2}} = \frac{b}{\sqrt{ac}} \; .$$

When a, b, c are known functions of η and τ, this expression will give the normalized correlation between v_1 and v_2 as a function of η and τ and will thus represent the normalized correlation function $S(\eta, \tau)/(\overline{v_1^2}\ \overline{v_2^2})^{1/2}$. In this way a connection between the probability and correlation functions is obtained.

It is necessary to point to another notion of importance connected with the function $\overline{\omega}$. This function, as mentioned, determines the simultaneous probability that v_1, v_2 will have certain specified values. The relative probability of v_2 having a specified value when v_1 is known, is determined by:

$$p(v_2; v_1) = \frac{\overline{\omega}}{\int_{-\infty}^{+\infty} \overline{\omega} \, dv_2}$$

If again we omit a constant factor and give attention to the exponential function only, the exponential occurring in $p(v_2; v_1)$ has the form:

$$e^{-b^2 v_1^2/c + 2bv_1 v_2 - c v_2^2} = e^{-c(v_2 - bv_1/c)^2} \; .$$

It must be kept in mind, however, that correct expressions for the relative probability function (like all other probability functions) must be defined in such a way that its integral over all values of v_2 from $-\infty$ to $+\infty$ is equal to unity.

One can also define a relative probability for v_1 if v_2 were known. There can be, however, reasons which make it more convenient or more natural to consider v_1 as a primary quantity and v_2 as a quantity which in some way is dependent on v_1; this is the case, for instance, when v_2 refers to a later instant than v_1.

We now go a step further and consider the possibility that there exists a probability function for the simultaneous occurrence of particular values of v at three different points P_1, P_2, P_3. This will express an interdependence between the velocities at these three points. As an example of such a probability function we take the Gaussian function:

$$\pi_1 = e^{-Av_1^2 + 2Bv_1v_2 + 2Cv_1v_3 - Dv_2^2 + 2Ev_2v_3 - Fv_3^2}$$

(again omitting a constant factor). The constants A, B, C, D, E, F in general will be functions of the coordinates of all three points. In the case of a homogeneous and stationary field they reduce to functions of the intervals between the points.

In an interesting application of such probability functions, to be discussed in the rest of this Appendix, it is of importance that the three points P_1, P_2, P_3 are arranged in a definite order so that there is no doubt that P_2 is between P_1 and P_3. This will be the case, for instance, when they refer to the same instant, so that they can be arranged according to increasing values of y; or when they refer to the same coordinate y and can be arranged according to increasing time. In particular, the latter case is useful since it makes it possible to imagine that v_2 in some way depends on v_1, and v_3 either on both v_1 and v_2, or on v_2 alone. We shall be concerned with the difference between these two possibilities.

The probability function π_1, given above, can again be applied for a number of calculations. We observe that if the exponent is reduced to a sum of squares (we omit the details), the integral of π_1 over all values of v_1, v_2, v_3 from $-\infty$ to $+\infty$ can easily be obtained; the result is:

$$J = \iiint_{-\infty}^{+\infty} \pi_1 \, dv_1 \, dv_2 \, dv_3 = \pi^{3/2} \Delta^{-1/2} .$$

Mean values can now be found with the aid of a similar formula as has been used before; for instance

$$\overline{v_1^2} = -\frac{1}{J}\frac{\partial J}{\partial A} .$$

When the calculations are carried out the results are:

$$\overline{v_1^2} = \frac{DF - E^2}{2\Delta} ; \quad \overline{v_2^2} = \frac{AF - C^2}{2\Delta} ; \quad \overline{v_3^2} = \frac{AD - B^2}{2\Delta}$$

$$\overline{v_1 v_2} = \frac{BF + CE}{2\Delta} ; \quad \overline{v_1 v_3} = \frac{BE + CD}{2\Delta} ; \quad \overline{v_2 v_3} = \frac{AE + BC}{2\Delta}$$

We may assume that the mean squares of v_1, v_2, v_3 are equal , in which case we must have:

$$DF - E^2 = AF - C^2 = AD - B^2 .$$

If this condition was not fulfilled originally, it can be assured by changing the scales for v_1, v_2, v_3. (For the deductions following below, it is not necessary that the condition be fulfilled.)

The normalized correlations between pairs of v's are given by:

$$\alpha = \frac{\overline{v_1 v_2}}{(\overline{v_1^2} \cdot \overline{v_2^2})^{1/2}} = \frac{BF + CE}{(AF - C^2)^{1/2} (DF - E^2)^{1/2}}$$

$$\beta = \frac{\overline{v_1 v_3}}{(\overline{v_1^2} \cdot \overline{v_3^2})^{1/2}} = \frac{BE + CD}{(AD - B^2)^{1/2} (DF - E^2)^{1/2}}$$

$$\gamma = \frac{\overline{v_2 v_3}}{(\overline{v_2^2} \cdot \overline{v_3^2})^{1/2}} = \frac{AE + BC}{(AD - B^2)^{1/2} (AF - C^2)^{1/2}}$$

One can now ask if it is possible that <u>the normalized correlation between v_1 and v_3 becomes equal to the product of the normalized correlations between v_1 and v_2, v_2 and v_3, respectively</u>. It should be observed that this is not a cyclic relation, and that its definition requires that v_2 shall be intermediate between v_1 and v_3 in some physical aspect. The condition requires $\beta = \alpha\gamma$ and comes down to the relation: $C\Delta = 0$. Since the expressions used presuppose that Δ is not zero, the only solution is: $C = 0$.

This result has important consequences, both with regard to the nature of the correlation function and with regard to the probability function. To take the latter subject first, it will be seen that now the probability function π_1 can be factored in such a way that one factor refers to v_1 and v_2, while the other factor refers to v_2 and v_3.

We observe that a probability function connecting three variables v_1, v_2, v_3 can always be used to obtain a probability function connecting two variables only, by integrating the function with respect to the omitted variable from $-\infty$ to $+\infty$. In this way we can obtain three separate probability functions, viz.:

connecting v_1, v_2: $\bar{\omega}_{12} = e^{-\frac{AF-C^2}{F} v_1^2 + 2\frac{BF+CE}{F} v_1 v_2 - \frac{DF-E^2}{F} v_2^2}$

connecting v_2, v_3: $\bar{\omega}_{23} = e^{-\frac{AD-B^2}{A} v_2^2 + 2\frac{AE+CD}{A} v_2 v_3 - \frac{AF-C^2}{A} v_3^2}$

and connecting v_1, v_3: $\bar{\omega}_{13} = e^{-\frac{AD-B^2}{D} v_1^2 + 2\frac{CD+BE}{D} v_1 v_3 - \frac{DF-E^2}{D} v_3^2}$.

In the case $C = 0$, the first two functions reduce to:

$$\bar{\omega}_{12} = e^{-Av_1^2 + 2Bv_1v_2 - (D - E^2/F)v_2^2}$$

$$\bar{\omega}_{23} = e^{-(D - B^2/A)v_2^2 + 2Ev_2v_3 - Fv_3^2} .$$

Now from $\bar{\omega}_{23}$ we deduce the relative probability for v_3 when v_2 is known and obtain (exponential factor only):

$$p(v_3; v_2) = \frac{\bar{\omega}_{23}}{\int_{-\infty}^{+\infty} \bar{\omega}_{23}\, dv_3} = e^{-(E^2/F)v_2^2 + 2E\, v_2v_3 - F v_3^2}$$

We then find the desired factorization of π_1 :

$$\pi_1 = \bar{\omega}_{12} \cdot p(v_3, v_2) .$$

To interpret the result we observe that as a general rule probability functions are multiplied when they refer to mutually independent probability relations. In the present case, when v_2 is known, there is a definite probability for v_3 which is completely independent of v_1. This probability is expressed either by the probability function $\overline{\omega}_{23}$ connecting v_2 and v_3, or by the relative probability function $p(v_3; v_2)$. An influence of v_1 on v_3 is only indirectly possible when v_2 is not known and when we must use the function $\overline{\omega}_{12}$ to find a probability for v_2 and then the function $\overline{\omega}_{23}$ to find the resulting probability for v_3. If it is assumed that all values are possible for v_2, there results a certain probability connecting v_3 with v_1 which is given by the function:

$$\overline{\omega}_{13} = \int_{-\infty}^{+\infty} \pi_1 \, dv_2 = \int_{-\infty}^{+\infty} \overline{\omega}_{12} \cdot p(v_3, v_2) \, dv_2$$

already mentioned before.

We observe that when the correct factors are inserted before the exponential functions the following relations hold:

$$\int_{-\infty}^{+\infty} p(v_3; v_2) \, dv_2 = 1 \qquad \text{for all } v_2$$

and:

$$\int_{-\infty}^{+\infty} \overline{\omega}_{13} \, dv_3 = \int_{-\infty}^{+\infty} \overline{\omega}_{12} \, dv_2 \quad .$$

The result expressed by the factorization of the function π_1 is often written in a slightly different form. Let us keep to the idea that P_1, P_2, P_3 represent three instants t_1, t_2, t_3 (following each other in that order) at the same point of the field. To make this explicit we write

$$p(v_2, t_2; v_1, t_1)$$

for the relative probability function for v_2 at the instant t_2, when it is known that the velocity had the value v_1 at the instant t_1. Such a relative probability function need not be based on the Gaussian function, but can also be used when the probability distribution has a different form. We always must have:

$$\int_{-\infty}^{+\infty} p(v_2, t_2; v_1, t_1)\, dv_2 = 1 \quad \text{for all } t_2, v_1, t_1,$$
$$\text{provided } t_2 > t_1.$$

A similar notation is used for the other relative probabilities. In the case considered, they are given by:

$$p(v_2, t_2; v_1, t_1) = \frac{\bar{\omega}_{12}}{\int_{-\infty}^{+\infty} \bar{\omega}_{12}\, dv_2}$$

$$p(v_3, t_3; v_2, t_2) = \frac{\bar{\omega}_{23}}{\int_{-\infty}^{+\infty} \bar{\omega}_{23}\, dv_3}$$

$$p(v_3, t_3; v_1, t_1) = \frac{\bar{\omega}_{13}}{\int_{-\infty}^{+\infty} \bar{\omega}_{13}\, dv_3} .$$

The result obtained can now be expressed by the formula:

$$p(v_3, t_3; v_1, t_1) = \int_{-\infty}^{+\infty} p(v_3, t_3; v_2, t_2) \cdot p(v_2, t_2; v_1, t_1)\, dv_2$$
$$\text{provided } t_3 > t_2 > t_1 .$$

Physical systems for which this relation between successive relative probabilities holds, whether they are based on a Gaussian distribution or not, are called "Markoff systems" or "Markoff chains". The property by which they are characterized is that the probability for the appearance of a particular value of the variable (in our case the velocity v) at a given instant t, is fully determined, when the value of v at one earlier instant is known. It will be evident that the interval between the two consecutive instants should not be too small; to express it somewhat roughly, the interval must be sufficiently long "to wipe out any influence from values still further back in the past". Nevertheless, in a group of mathematical deductions concerning probability functions of the type considered here, directed towards the construction of differential equations, it is assumed that the time interval may become infinitely small. This, in a way, establishes a conflict between physical relations and the application of mathematics to represent them and to make them amenable to calculations; careful reasoning is required to find out how far one may go with the mathematical assumption. This point, however, will be left aside here and we will not go into these differential equations.

We must still consider the meaning of our result for the correlation function. We had assumed that the normalized correlation function between the values of v at the instants P_1 and P_3 was equal to the product of the normalized correlations between the values at P_1 and P_2, and P_2 and P_3, respectively. When the normalized correlation function is dependent on the time interval only, we can express this result by means of the formula:

$$\text{norm. corr. }(\tau_1+\tau_2) = \text{norm. corr. }(\tau_1) \times \text{norm. corr. }(\tau_2)\,,$$

where $\tau_1 = t_2 - t_1$; $\tau_2 = t_3 - t_2$; and $\tau_1 + \tau_2 = t_3 - t_1$.

Such a relation can only be assured if the normalized correlation function is a simple exponential function of the interval:

$$\text{norm. corr. }(\tau) = e^{-\alpha\tau}\,, \quad \text{or rather } e^{-\alpha|\tau|}$$

since it can depend only on the absolute value of the interval τ. Here α is a constant independent of τ, which is not determined by the product rule, but which must come from other data.

The formula satisfies the condition that the normalized correlation becomes unity when the interval is zero. The formula does not satisfy the condition that the derivative with respect to τ should be zero for $\tau = 0$. It can, therefore, not be correct down to $\tau = 0$. This means that some of the relations we have used, in particular the product relation, cannot be true for extremely small intervals. We have come to a similar limitation in connection with the definition of Markoff systems. The relations considered here consequently can be valid only for intervals (time intervals, or other intervals) exceeding a certain threshold, but with this restriction they are of great importance for certain physical systems.

We shall see in Chapter IV that a correlation function of the exponential type mentioned above is characteristic for a certain aspect of the turbulent fluctuations of the velocity.

CHAPTER III

The Spectrum of a Turbulent Field

15. We have seen that the Eulerian correlation function gives information about connections in the field of flow extending over space and time. This brings us to the problem of the structure of the field at a given instant. Various methods are available for analyzing the geometrical pattern of the motion. We shall consider two of them.

For simplicity, we consider a turbulent field without mean motion and assume the turbulence to be homogeneous in space. It does not make much difference if we start with a three-dimensional field, but in further work it is simpler to restrict ourselves to the one-dimensional case and to postpone consideration of the three-dimensional fields.

One method (applied by von Weizsäcker and others) starts from the idea of taking average values of the velocities over domains of decreasing dimensions. We imagine a scale of lengths L_0, L_1, L_2, L_3 ....., in which L_0 is large compared with the largest eddies which observation may disclose in the field, while every succeeding term of the series is obtained by taking 1/2 of its predecessor. We now associate with every point x, y, z of the field a series of velocities U_n, V_n, W_n, obtained by averaging the values of the actual velocities around this point over a volume of magnitude L_n^3, having its center at x, y, z. It can be expected that when L_0 is large enough, the values of U_0, V_0, W_0 will be zero for every point x, y, z, so that averaging over a volume L_0^3 obliterates all details from the field. When we go down the scale, results will be obtained which differ from zero, and gradually more and more details of the field will become apparent so that the picture arrived at will become more and more precise. This will not go on indefinitely. We must expect that when we have come to a certain small length, say L_N, depending on the nature of the field but in all normal cases large compared with molecular distances, a further subdivision will not reveal new details. This means that there is a certain minimum scale in the pattern of motion, so that the motion shows complete coherence over distances of the order L_N or smaller, which still remains far from the molecular field. The presence of a "gap" between the pattern of the hydrodynamic field and the molecular structure was already pointed out by Osborne Reynolds in his classical paper on turbulence. It was considered by him as a cornerstone of great importance in the analysis.

It can be observed that the minimum scale mentioned here is connected with the microscale λ defined by Eq.(12) of the preceding chapter. Actually L_N must be smaller than λ .

All these calculations of averages refer to one single instant of time. The analysis can be performed for a series of instants of time, so that at every point of the field we may obtain U_n, V_n, W_n as functions of the time.

We now introduce a series of "component fields" defined by the formulas

$$u_n = U_n - U_{n-1}, \quad v_n = V_n - V_{n-1}, \quad w_n = W_n - W_{n-1}.$$

This means that the field u_n, v_n, w_n gives the details which were not yet apparent in U_{n-1}, V_{n-1}, W_{n-1}, but which are given by U_n, V_n, W_n . Every component field thus adds further details to the picture.

From what has been said concerning the behavior of U_n, V_n, W_n for large n (in particular for $n > N$), it follows that the component fields u_n, v_n, w_n become zero for $n > N$.

If further we define:

$$u_o = U_o, \quad v_o = V_o, \quad w_o = W_o,$$

we can write for any n:

$$U_n = u_n + u_{n-1} + u_{n-2} + u_{n-3} + \cdots \quad + u_o, \text{ etc.}$$

In this way we can say that the field of motion, as observed to any prescribed degree of precision, is built up from component fields u_n, v_n, w_n, each component, as mentioned, adding a certain amount of detail.

All these calculations of averages and the analysis built upon them refer to one single instant of time. The analysis can be performed for a series of instants, and in this way we can obtain U_n, V_n, W_n, or u_n, v_n, w_n, as functions of the time.

The analysis of the field can now be used as a basis for a description of various characteristic features of the turbulent motion. One of these features is the kinetic energy of the motion. Here, however, we encounter a difficulty. We can calculate the mean kinetic energy for each of the component fields by integrating the quantity

$$\frac{1}{2}\rho(u_n^2 + v_n^2 + w_n^2)$$

over the field. However, one cannot, in general, expect that the mean kinetic energy for the actual field can be obtained by summing the energies of the component fields. This will only be the case if all integrals of the type

$$\iiint (u_n u_m + v_n v_m + w_n w_m)\ dx\,dy\,dz$$

would be zero, for every n and m. There may be fields for which this is (either exactly, or approximately) true, but it is not easy to state which conditions must be satisfied in order to obtain this result. In most cases the result will not apply.

One could calculate the mean kinetic energy for each field U_n, V_n, W_n, and then take the differences of the various values as energies to be associated with the component fields, but this procedure is not satisfactory either. Perhaps the method indicated first may even give a better approximation. It can be used at any rate to obtain a provisional picture of the distribution of the kinetic energy over the component fields, that is of the "spectrum" of the given turbulent motion.

It is for this reason that in mathematical work on the structure of the turbulent field preference is given to other methods of analysis, based on the application of a system of so-called "orthogonal functions". These functions satisfy the property that integrals of the type considered above, that is, over the product of two component fields, exactly vanish. The method most in use is that of the Fourier analysis.

16. Application of Fourier analysis. Restricting to a single coordinate, for which we will take y, we represent the velocity $v(y, t)$, in the Eulerian description, for a given instant t by means of a Fourier integral:

$$v = \int_{-\infty}^{+\infty} \varphi(k)\ e^{iky}\ dk \tag{1}$$

where $i = \sqrt{-1}$. The function $\varphi(k)$ is called the amplitude function; it will be a complex function which must satisfy the relation $\varphi(-k) = \varphi^*(k)$, the asterisk denoting the complex conjugate, since otherwise v would not be a real function. The amplitude function will, of course, be a function of the time. The equations of this section and of the next one, however, refer to a single instant of time.

The introduction of a Fourier integral brings a new difficulty. Fourier integrals can be defined only for functions which vanish at infinity in such a way that their squares admit a finite integral. Since we are considering turbulence homogeneous in space, the statistical properties of v must be the same in every domain of the y-axis, whether this is near to the origin or far away. Consequently $\overline{v^2}$ will have a mean value which is the same everywhere, and its integral will be divergent.

To overcome this difficulty one usually assumes that we can restrict the analysis to the values of v within a finite domain, say within the part of the y-axis from $-M$ to $+M$, while outside this domain v is replaced by zero. The corresponding Fourier integral will now give the values of v within this domain only, but if M is large, the integral expression will be sufficiently representative. The amplitude function $\varphi(k)$ is then obtained from the inverse formula:

$$\varphi(k) = \frac{1}{2\pi} \int_{-M}^{+M} v(y)\, e^{-iky}\, dy. \tag{2}$$

For the proof of this relation we must refer to textbooks on Fourier analysis.

Before discussing the expression for the kinetic energy connected with formula (1), we will consider the correlation function.

17. Formula (1) can be applied to calculate the Eulerian correlation function $S_1(\eta) = \overline{v_1(y)\, v_2(y+\eta)}$, as defined by Eq. (5), Chapter II, p. 24. We form:

$$v_1 v_2 = \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} dk''\ \varphi(k')\ \varphi(k'')\ e^{i(k'+k'')\,y + ik''\eta}.$$

This expression will be valid so long as both y itself and $y + \eta$ are within the domain from $-M$ to $+M$. Since we can expect that the correlation will become to zero, when η exceeds a certain limit, we can restrict to the considerations of values of η below this limit. If then we choose M very much larger than this limit, we do not make a great error if we say that the expression is valid when $-M < y < +M$. Consequently, we can obtain the mean value $\overline{v_1 v_2}$ by integrating the expression over y from $-M$ to $+M$ and dividing the result by $2M$. Actually we take the integration limits as $-M_1$ and $+M_1$ where $M_1 > M$. This can add nothing to the integral; hence to obtain the mean value we must still divide by $2M$. The application of wider limits in the integration makes it possible to introduce a limiting process which considerably similifies the result.

The integration gives:

$$v_1 v_2 = \frac{1}{M} \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} dk'' \, \varphi(k') \, \varphi(k'') \, e^{-ik''\eta} \, \frac{\sin(k'+k'')M_1}{k'+k''} .$$

For $M_1 \rightarrow \infty$ this transforms into:

$$\frac{\pi}{M} \int_{-\infty}^{+\infty} dk' \, \varphi(k') \, \varphi(-k') \, e^{-ik'\eta} .$$

For the proof we again refer to textbooks. Since in this integral we may just as well write k for the integration variable instead of k', we obtain:

$$v_1 v_2 = \frac{\pi}{M} \int_{-\infty}^{+\infty} dk \, \varphi(k) \varphi(-k) \, e^{-ik\eta} = \frac{2\pi}{M} \int_{0}^{\infty} dk \, \varphi(k) \, \varphi^*(k) \cos k\eta .$$

We now write:

$$(3) \qquad \Gamma(k) = \frac{2\pi}{M} \varphi(k) \, \varphi^*(k) .$$

The correlation function can then be represented by means of the integral:

$$(4) \qquad S_1(\eta) = \overline{v_1 v_2} = \int_0^{\infty} \Gamma(k) \cos k\eta \, dk.$$

In connection with Eq. (3) it is of importance to observe that the amplitude function $\varphi(k)$ depends on M, since M occurs in the limits of the integral (2). On the other hand, the correlation function S_1 by its nature must be independent of M, and the same must apply to the function $\Gamma(k)$. It follows that the absolute value of the amplitude function must be proportional to $M^{1/2}$. According to (2), the value of $\varphi(k)$ is obtained by addition of a large number of vectors $v(y)\, e^{-iky}$, in which $v(y)$ varies incessantly from positive to negative values and vice versa. The "number of these vectors" is not a quantity which can be precisely defined, but we may certainly assume that it is proportional to M. The fact that the resulting vector has a length proportional to $M^{1/2}$ is then related to similar results obtained in calculating sums of a large number of random positive and negative contributions, as for instance occurs in the problem of the "random walk". However, the only exact proof is obtained by calculating $\varphi(k)$ from formula (3), supposing that the correlation function $S_1(\eta)$ is known and that $\Gamma(k)$ is deduced from the inversion of (4), which has the form:

$$\Gamma(k) = \frac{2}{\pi} \int_0^\infty S_1(\eta) \cos k\eta \, d\eta \, . \tag{5}$$

It will be seen that formula (3) can only give the absolute value of $\varphi(k)$ and gives nothing about the phase angle of this complex quantity. There is no information about the phase angle, unless we are able to calculate φ directly from the integral (2), and it may be that the phase angle behaves very irregularly when k changes, but there is more to be observed. For convenience we took the limits in (2) as $-M$, $+M$. However, the mean value of the correlation function $S_1(\eta)$ must not change if we shift the integration interval along the y-axis, and use limits $y_o - M$, $y_o + M$, where y_o may be any value. Hence the absolute value of $\varphi(k)$ must be insensitive to such a shift, but it is probable that the phase angle will be influenced and it may, perhaps, fluctuate very wildly when y_o goes through a series of values.

When in formula (4) we take $\eta = 0$, we obtain the mean square of the velocity $\overline{v^2}$. Passing to the mean kinetic energy and omitting for simplicity the density factor, we find:

$$E = \frac{1}{2} \int_0^\infty \Gamma(k)\, dk \, . \tag{6}$$

This formula shows that with the aid of the Fourier analysis we have succeeded in resolving the mean kinetic energy of the actual field per unit volume into separate contributions, each referring to a definite Fourier component of (spatial) frequency k.

It will be seen that this resolution can be obtained without a direct calculation of the amplitude function $\varphi(k)$ from formula (2), which, in general, would be a practical impossibility. It is sufficient when we know the correlation function $S_1(\eta)$ for the field, either from experimental research or because it may be the only statistical datum concerning the field which has been given to us. From $S_1(\eta)$ we deduce $\Gamma(k)$ by means of formula (5), and this $\Gamma(k)$ gives us what is called the energy spectrum of the turbulence, defined with reference to a resolution of the field into spatial components.

18. The analysis of the preceding sections referred to harmonic components in space (complex exponential, or goniometric, functions of ky). When the turbulence is stationary with respect to time, we can make a similar resolution of the velocity at any point of the field, as a function of the time.

In order not to burden the notation too much, we particularly use the same letters as before and replace formula (1) by:

$$(7) \qquad v = \int_{-\infty}^{+\infty} \varphi(\omega)\, e^{i\omega t}\, d\omega ,$$

where the amplitude function $\varphi(\omega)$, which is a different function from that occurring in (1), is now defined with respect to frequencies ω in time. The expression will be valid for a certain length of time, which we can denote by $-T < t < +T$. If we replace (3) by:

$$(8) \qquad \Gamma(\omega) = \frac{2\pi}{T}\, \varphi(\omega) \quad (\omega^*)$$

(with, of course, a new function Γ), we obtain the following expression for the Eulerian time-correlation function $S_2(\tau)$:

$$(9) \qquad S_2(\tau) = \overline{v_1 v_2} = \int_0^{\infty} \Gamma(\omega) \cos \omega\tau \, d\omega .$$

The expression for the mean kinetic energy per unit volume becomes:

$$(10) \qquad E = \frac{1}{2} \int_0^\infty \Gamma(\omega)\, d\omega .$$

It is the resolution in time which is most easily performed in the experimental analysis of turbulent motion. If we change from $S_2(\tau)$ to the normalized correlation function $S_2(\tau)/\overline{v^2}$ and instead of $\Gamma(\omega)$ introduce $F(n) = \Gamma(\omega)/2\pi\overline{v^2}$, where $n = \omega/2\pi$, we find:

$$(10a) \qquad 2E = \overline{v^2} \int_0^\infty F(n)\, dn .$$

19. Additional observations. The method of resolving the velocity field with regard to frequencies in space can be used whenever the field is homogeneous. The resolution can then be made at any definite instant; the spectrum obtained will refer to that same instant. When the turbulence is decaying, it is possible to describe the decay by following the gradual change of the spectrum with time. It can be found that certain components disappear rapidly, whereas others may change only very gradually and perhaps some of them may first show a temporary increase.

The method of resolving the velocity with regard to frequencies in time can be used whenever the field is stationary. The resolution can be made for any point of the field, and it is possible that the spectrum will change when we go from one point to another. For instance, in considering boundary layer turbulence, the spectrum will change when we approach the wall of the field. In the case of wind tunnel turbulence produced by a grid, which is stationary in time and which is only approximately homogeneous in space, there will be a gradual change in the spectrum when we go further downstream, that is, away from the grid.

When the field is both homogeneous and stationary, the spatial spectrum will be independent of the time and the time spectrum will be the same for all points. In principle it is then also possible to make a resulution by means of a double Fourier integral:

$$(11) \qquad v = \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \; f(k,\omega)\, e^{i(ky+\omega t)} .$$

The representation shall be valid for $-M < y < +M$; $-T < t < +T$; outside of this domain it will give $v = 0$. We can then form the complete Eulerian correlation function and obtain:

$$S(\eta,\tau) = \frac{\pi^2}{MT} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega \; f(k,\omega)\, f(-k,-\omega)\, e^{-i(k\eta+\omega\tau)} .$$

If we introduce the functions:

$$F(k,\omega) = \frac{2\pi^2}{MT} f(k,\omega)\, f(-k,-\omega); \quad G(k,\omega) = \frac{2\pi^2}{MT} f(k,-\omega)\, f(-k,\omega),$$

this can also be written:

$$(12) \quad S(\eta,\tau) = \int_{0}^{\infty} dk \int_{0}^{\infty} d\omega \left\{ F(k,\omega) \cos(k\eta+\omega\tau) + G(k,\omega) \cos(k\eta-\omega\tau) \right\} .$$

The former resolutions can be obtained from this general formula by carrying out either the integration with respect to ω or the integration with respect to k.

It must be observed that in defining the spectrum, sometimes another point of view is taken. The Fourier analysis, say for the value of v as a function of the time at a given point of space, is made repeatedly over a set of consecutive periods of equal large duration T. Since we now work with finite periods, we can use a Fourier series instead of the integral, so that the amplitude function is replaced by an (infinite, but enumerable) set of amplitude coefficients, as follows:

$$v(t) = \Sigma \left(a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right) .$$

For each period T, a new set of coefficients is obtained. Consequently, when we consider the coefficients a_n, b_n of the n-th component, they will show a collection of values, which can be described by a probability function if the collection is large enough. In this way one arrives at the notion of a probability function for the spectral amplitudes.

Evidently there must be a relation between such a probability function and the character of the amplitude function $\varphi(k)$ when the Fourier integral is used, but the exact formulation of such a relation will be no simple matter.

The notion of a probability function for the amplitudes can also be brought forward if we consider an ensemble of systems and to each of them apply the Fourier analysis. We shall, however, not go into these problems.

CHAPTER IV

Some Experimental Data on the Spectrum and on Correlation

20. Data on the spectrum referring to frequencies in time.* Experimental investigations have mostly been made for the case of grid-produced wind tunnel turbulence, where we have to do with turbulence decaying in time, swept along by the general current in the wind tunnel (which has a constant mean velocity). For a measuring instrument fixed in space, the field is statistically stationary. While the field is actually three-dimensional, the measuring instrument practically follows one component only in its dependence on the time at a fixed point, say $u(t)$. The corresponding frequency spectrum $F(n)$ is defined as the fraction of $\overline{u^2}$ contained in the (time) frequency band n, $n + dn$, and is often called the "power spectrum". The corresponding Eulerian time-correlation function, in our notation: $S_2(\tau)$, is then given by:

$$S_2(\tau) = \overline{u^2} \int_0^\infty dn\, F(n) \cos 2\pi n\tau \; .$$

Sometimes a function $\Psi(\tau)$ is used, defined by:

$$\Psi(\tau) = \frac{S_2(\tau)}{\overline{u^2}} = \int_0^\infty dn\, F(n) \cos 2\pi n\tau \; .$$

Dryden** already found that the measurements of $F(n)$ for a certain range of frequencies can approximately be represented by the function

$$\text{(a)} \qquad F(n) \cong \frac{A}{1 + c^2 n^2} \; .$$

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*In preparing this section use has been made of a report on the Spectrum of Isotropic Turbulence by H. W. Liepmann, J. Laufer and Kate Liepmann.

**H. L. Dryden, "A Review of the Statistical Theory of Turbulence," Quart. Appl. Mathem. 1, p. 35, 1943.

If this is used up to $n \rightarrow \infty$ the corresponding correlation function becomes:

$$\text{(b)} \qquad \Psi(\tau) = e^{-2\pi\tau/c} .$$

Formula (a), however, cannot be valid for large values of n. Nevertheless, Liepmann's measurements showed that by far the largest part of the turbulent energy is contained in a portion of the spectrum for which the simple expression gives a good approximation.

It is probable that the formula will not be valid down to $n = 0$, since finite values of $F(n)$ down to $n = 0$ would seem to imply the presence of correlations extending to indefinitely increasing time intervals τ. Experimentally, most measuring appliances cannot go far below frequencies of 1 per second, so that no reliable data are available for periods of a minute, or say an hour or more.

21. Other experiments have shown that the probability distribution of u(t), referring to its instantaneous values without giving any attention to correlation, is Gaussian, so that the probability of $u(t)$ having a value between u and $u + du$ is given by:

$$\text{(c)} \qquad \frac{1}{\sqrt{2\pi\,\overline{u^2}}}\, e^{-u^2/2\overline{u^2}}\, du.$$

It is proved in papers on the theory of the Brownian movement that a process which gives a Gaussian distribution for instantaneous values, and which has a correlation function of the simple exponential type (b), mentioned above, must be a "Markoff" random function. Such a function is characterized by the following property: suppose that we know the value of u at an instant t_o; let it be u_o. The probability distribution of the values at a later instant t is then completely determined, so that it can be described by a function $P(u, t;\ u_o, t_o)$. Data concerning the value of u at instants before t_o are irrelevant. The function $P(u, t;\ u_o, t_o)$ therefore is the full expression of all that we can say about correlation in the behavior of $u(t)$.

The function $P(u, t;\ u_o, t_o)$ must satisfy an important condition. If the value of u at t_o is given, and we desire to know the probability distribution for the values at t, we can choose an intermediate instant t_1 and first write down the probability distribution for that instant:

$$P\,(u_1, t_1;\; u_o, t_o)\,.$$

The probability distribution for the instant t will then follow from the integral

$$\int_{-\infty}^{+\infty} P\,(u, t;\; u_1, t_1)\;\; P\,(u_1, t_1;\; u_o, t_o)\, du_1\,.$$

The value of this integral must be identical with $\underline{P}\,(u, t;\; u_o, t_o)$.

It will be evident that the interval between the two consecutive instants ($t - t_o$; or $t - t_1$ and $t_1 - t_o$) in this reasoning may not be taken too short. Otherwise one could expect that the value at the instant t might be found with reasonable accuracy from the value at t_1 together with the rate of change at t_1, which actually would mean that we have made use of data concerning two preceding instants. If, however, $t - t_1$ exceeds a certain threshold, data concerning the rate of change of u at the instant t_1 can be irrelevant for the prediction of future values, which means that the course of $v(t)$ can change in an unknown way between t_1 and t.

It is important to observe that a correlation function of the simple exponential type (b) is characteristic for processes in which the variable quantity changes abruptly with a finite amount at random instants of time. The mean interval between consecutive changes is related to the parameter c in the exponent. The abruptness of the change is evident from the fact that $\Psi(\tau)$ has a nonzero derivative for $\tau = 0$. Actually, this is impossible in any real physical process. A correlation function for a physical quantity must always present a rounded top (with horizontal tangent and nonzero radius of curvature) at $\tau = 0$ and the spectral function $F(n)$ must decrease much more rapidly for large n than is indicated by the approximation (a).

With reference to turbulence, this means that an important part of the phenomenon proceeds as if elements of volume with different values of the velocity u follow each other in an irregular way, with very thin transition regions separating them. When a transition region sweeps over the measuring instrument, the rate of change of u is very high; in between the transition regions the rate of change is low. Closer inspection will reveal that the transition regions have a certain (very small) finite thickness; $\partial u/\partial t$ does not exceed a certain order of magnitude and $F(n)$ probably will decrease as e^{-Cn} for $n \rightarrow \infty$, C being some constant.

22. It will not be strange that, although the probability function for $u(t)$ is Gaussian, as indicated in (c), the corresponding function for $\partial u/\partial t$ is not Gaussian. A measure for the deviation is given by the dimensionless quantity:

$$T = \frac{\overline{(\partial u/\partial t)^4}}{\left(\overline{(\partial u/\partial t)^2}\right)^2}$$

For a Gaussian distribution one would have $T = 3$. Actual values are larger. Townsend* gives values ranging from 3,32 to 3,49, and Batchelor,** in a later work, assumes the value 3,5. For u itself, Townsend mentioned values ranging from 2,92 to 2,99, fairly in agreement with the theoretical value 3.

There even is no symmetry between positive and negative values of $\partial u/\partial t$. Although the mean value of this quantity is zero, the mean value of the third power $(\partial u/\partial t)^3$ in general appears to differ from zero. A dimensionless measure of the deviation is the skewness factor:

$$S = \frac{\overline{(\partial u/\partial t)^3}}{\left(\overline{(\partial u/\partial t)^2}\right)^{3/2}}$$

for which the value $-0,39$ has been obtained with isotropic turbulence (Batchelor, l.c.).

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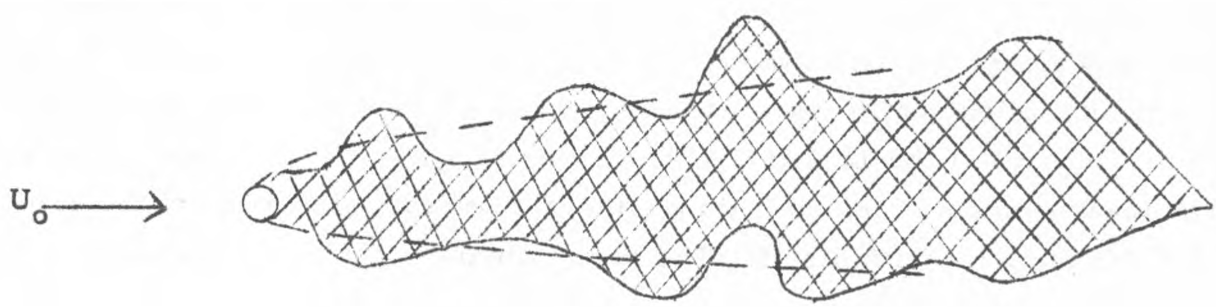
*A. A. Townsend, "Correlation Derivatives in Isotropic Turbulence," Proc. Cambridge Philos. Soc. 43, p.560, 1947.

**G. K. Batchelor, "Recent Developments in Turbulence Research," Proc. VIIth Congress of Appl. Mechanics, London, 1948.

Both the almost abrupt changes in u and the skewness presented by its derivative are due to the nonlinear character of the hydrodynamic equations, to which we shall come back later. (The nonlinear effects primarily produce a skewness in $\partial u/\partial x$; in the experimental setup this is measured as $(1/U_o)(\partial u/\partial t)$).

As a particular form of turbulence, we mention the turbulence in the two-dimensional wake behind a single cylinder. This may be considered as the elementary form compared to which the grid-produced turbulence in a wind tunnel is a combination and superposition of a great many wakes with different planes of symmetry. In the wake behind a single cylinder there appears a strong intermittent type of velocity fluctuation. For a certain fraction of the time the velocity varies in the ordinary irregular manner, as is characteristic for well developed turbulence, while for the remainder of the time the velocity fluctuations are slow and of small magnitude*. It looks as if there is a rather sharp division of the flow field into a laminar flow outside a wholly turbulent wake core with an irregular boundary, while within the core the turbulence has small-scale isotropy. At any point of the wake, laminar and turbulent flow will occur intermittently as the irregularities of the core are carried downstream.



*G. K. Batchelor, "Note on Turbulent Free Flows," Journn. Aero. Sciences 17, p. 44, 1950.

23. The experimental results considered thus far have been obtained with fixed measuring instruments, and consequently refer to the Eulerian description. In deriving certain formulas connected with the results, it even has been assumed by the authors that the Lagrangian derivative du/dt would be small compared with the local Eulerian derivative $\partial u/\partial t$, and that one might approximately write $\partial u/\partial t \cong -U_0(\partial u/\partial x)$, U_0 being the constant velocity of the general current in the wind tunnel. Hence the results deduced from these measurements do not give light on Lagrangian correlation; the assumption made entails only that it shall stretch over much larger intervals of time than the local Eulerian correlation $S_2(\tau)$.

Measurements referring to the Lagrangian point of view have been made by following the motion of individual particles carried along by the fluid. When the particles have a density equal to that of the liquid and are small compared with the size of the eddies which form the main part of the turbulence, it can be presumed that they will closely reproduce the motion of the elements of volume of the fluid. For particles starting from a fixed point in a flowing stream (either a wind tunnel current or a current in a canal with boundary layer turbulence), measurements have been made concerning their lateral displacements after they have moved downstream over a certain distance. These measurements have given the result that the transverse displacements are distributed according to a Gaussian curve. Further, the increase of the mean square transverse displacement with distance downstream, which approximately means increase with time elapsed since a particle was ejected from its orifice, permits calculation of the diffusion coefficient D considered in Section 10. This gives the time integral of the Lagrangian correlation function $R_v(\tau)$. More detailed measurements for small time intervals permit calculation of the correlation function $R_v(\tau)$ itself. The result obtained could be approximately represented by

$$R = e^{-\tau U_0/a},$$

where a is a constant.*

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*A. A. Kalinske and C. L. Pien, "Eddy Diffusion", Industrial and Engineering Chemistry 36, p. 220, 1944.

This result seems to indicate that the motion of the elements of volume of the fluid is likewise subject to almost abrupt random changes in velocity.

24. No data are known thus far, giving coexisting curves for the Eulerian and the Lagrangian correlation functions, referring to the same case of turbulent motion. It is very desirable that such measurements be made.

In the present case, where we consider particles carried along by a stream of approximately constant velocity, the mathematical relations are different and somewhat simpler than those considered in Sections 13 and 14. If the general motion is in the x-direction and again has the velocity U_o, we may consider the transverse velocity v and the transverse displacement y of a particle as functions of the time t and of the dowstream position x of the particle. We can then consider Eulerian correlation between particles observed at the same instant of time at various values of x, and thus determine a correlation function for <u>the transverse velocities</u>

$$S_1(\xi) = \frac{\overline{v(x,t)\ v(x+\xi,t)}}{\overline{v^2}} ,$$

where ξ is the difference of the x-values. We now divide all correlation functions by the mean square velocity $\overline{v^2}$, in order to have normalized expressions which take the value unity for $\xi = \tau = 0$. Or we may consider all particles passing through a fixed plane (x = constant), and determine a correlation function

$$S_2(\tau) = \frac{\overline{v(x,t)\ v(x,t+\tau)}}{\overline{v^2}} ,$$

depending on the time difference. Finally, we can determine a general Eulerian correlation function

$$S(\xi,\tau) = \frac{v(x,t)\ v(x+\xi,\ t+\tau)}{\overline{v^2}} .$$

by comparing particles having time and position differences. This function can be represented by lines of constant S in a diagram having ξ and τ as coordinates.

If we follow the history of every single particle, then also a Lagrangian correlation function can be determined referring to the life history of a particle, or, in other words, to the life history of a single element of volume of the fluid. We shall write for this function

$$R_v(\tau) = \frac{\overline{v(t)\ v(t+\tau)}}{\overline{v^2}} .$$

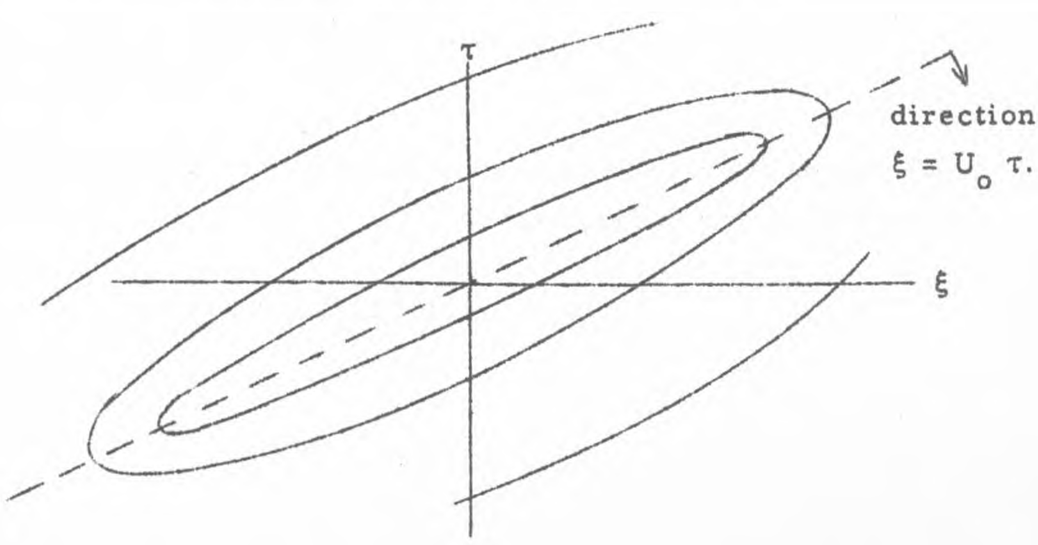
the two values of v referring to the same particle.

In the case considered, the Lagrangian time derivative of the transverse velocity is approximately given by

$$\frac{dv}{dt} \cong \frac{\partial v}{\partial t} + U_o \frac{\partial v}{\partial x} .$$

To the same approximation the Lagrangian correlation function $R_v(\tau)$ will approach to the Eulerian correlation function $S(\xi, \tau)$, if we take $\xi = U_o\tau$ in the latter.

If the Lagrangian time derivative is small in comparison with the local Eulerian time derivative $\partial v/\partial t$, the Lagrangian correlation function $R_v(\tau)$ will decrease from its maximum value unity at a much slower rate than the Eulerian time correlation function $S_2(\tau)$. We may then expect that the diagram for the function $S(\xi, \tau)$ will be somewhat of the shape:



25. Some General Observations on Experimental Data. By means of the hot-wire technique, Eulerian mean values and correlations can be obtained as time mean values in a stationary field, with measuring apparatus held in a fixed position. We mention that, for instance, the following quantities can be measured: the behavior of the longitudinal velocity component u and of the transverse components v and w as functions of time; the mean squares of these components and their spectra; correlation products referring either to a single component or to two different components measured simultaneously at different points of the field; the correlation product for a single component in which the two values refer to different instants of time measured at the same spot. (This requires recording the component and reading the record by means of two instruments which pick up the values for two instants with a prescribed interval of time between them.)

It is also possible to measure directly triple correlation products, in which the square of one component is multiplied by the first power of a component measured at a different spot.

Further, one can obtain the local time derivatives of the components and measure their mean square values and analyze their spectra. Also, mean square values of higher derivatives can be obtained.

Lagrangian correlations can be obtained only by the observation of particles carried along by the fluid.

The two types of turbulence for which most measurements have been made are grid-produced turbulence in a wind tunnel or the like, and the turbulent motion in a tube or canal, or in a boundary layer.

In the first case the field is stationary; i.e., over a large space its average structure is independent of the transverse coordinates y and z and it is slowly changing with x.

In the case of motion through a tube or channel with cylindrical or prismatic walls, the field will be stationary and independent of x. There is, however, a strong dependence on the transverse coordinate which is connected with a transmission of energy across the field to compensate the dissipation. In the experimental investigation of the field one must give due attention to the fact that it is not homogeneous. This will be felt in particular when observations are made on the motion of particles in order to obtain Lagrangian correlations, since the motion transverse to the general direction of the stream will bring them into regions where the state of the field is different.

It should not be forgotten that there are many more cases of turbulence apart from the two mentioned, several of which likewise are of technical importance; for instance, the problem of <u>mixing</u>. Suppose that in a closed space a fluid is at rest; a certain motion is introduced by stirring, etc. How shall one obtain a complete mixing of the fluid in such a way that admixtures introduced locally will become distributed evenly over the whole mass? One may consider either continued stirring, or an initial period of stirring, after which the field is left to itself. In such a case the motions do not directly interest us except insofar as it costs energy to produce them; what is of practical interest is the distribution of admixed material. In the case of an electrolyte added to the water, measurement of the electric conductivity as a function of the time at a number of spots can give a picture of the degree of homogeneity obtained and of the time needed for obtaining it.

The mathematical problem appearing here has some resemblance to the so-called ergodic problem. Can one expect that a given element of the fluid will follow a path which will practically bring it to every region of the field? To make the problem more precise, we consider the motion of a particle of the same density as the fluid; further, we divide up the field into cells of equal volume ω and ask whether the particle will ultimately pass through all these cells or whether there will be a preference for a certain group of cells, whereas others perhaps might not be reached, or only very infrequently. The volume given to the cells will influence the result; (technical points of view may sometimes determine a convenient size, not too small).

When, instead of the particle, we consider some dissolving or diffusing substance, ordinary diffusion or perhaps small-scale turbulence may ultimately bring about homogeneity.

This type of problem brings us still further from the realm of the Eulerian correlations than the Lagrangian correlation problem did-- the more so since the quasi-ergodic problem presents itself not only with fields which, in the main, are stationary, but also with fields which gradually damp out. Since large scale eddies may have long lifetimes, they could promote mixing efficiently perhaps long after the motion had been initiated.

26. Physical Interpretation of the Relation between the Spectrum and the Correlation Function. In deducing Eq. (3), Section 17, for the spectral function $\Gamma(k)$ giving the distribution of the energy of the harmonic components of the field, we made use of a mathematical artifice in order to get around certain difficulties connected with Fourier integrals. It may be useful to consider the experimental determination of the spectrum in order to see why a similar difficulty is not encountered there. We take the case of a function of the time, since this is the more common problem, which can be handled much more easily than would be the case with the spatial spectrum of a function of the coordinates. The simplest example refers to the velocity $u(t)$ as measured with the aid of a hot-wire anemometer at a fixed point of the field.

The electrical signal obtained from the hot-wire anemometer, after having been amplified, is passed through an electric filter adjusted so as to transmit only frequencies within a band of limited width. The transmitted signal can be applied to a thermo-cross, by means of which its mean square can be found. Now an electric filter is a combination of inductances, capacities and resistances. With the incoming electric signal $u(t)$, we shall write $w(t)$ for the outgoing signal. The relation between them can be calculated from the circuit and is usually expressed by means of a differential equation; for instance:

$$\frac{d^2w}{dt^2} + 2p\omega\frac{dw}{dt} + \omega^2 w = a\frac{du}{dt},$$

where ω, p and a are quantities depending on the circuit. It has been supposed that the incoming signal operates through induction, so that what really comes in is its time derivative.

In the particular case where the incoming signal would be a pure harmonic wave:

$$u(t) = A e^{int}$$

we find:

$$w = \frac{ian A e^{int}}{(\omega^2 - n^2) + 2ipn\omega} .$$

Hence, if p is small (actually we take p to be small compared with unity), the only frequencies which are transmitted through the filter are those which differ only slightly from ω. We can say that ω determines the center of the band of frequencies which can pass through the filter, while the band width appears to be proportional to $p\omega$. If we take $n = \omega$, the expression reduces to:

$$w = \frac{a}{2p\omega} A e^{int}$$

Hence in order to have a constant scale factor, one must make a proportional to ω.

When an arbitrary time function $u(t)$ is used as incoming signal, the outgoing signal is given by the integral:

$$w = \frac{a}{\sqrt{1-p^2}} \int_0^\infty dt_1 \, u(t-t_1) \, e^{-p\omega t_1} \cos(\omega t_1 \sqrt{1-p^2} + \epsilon) ,$$

where ϵ is defined by $\sin \epsilon = p$. We use this integral to calculate the mean value of $\overline{w(t)\, w(t+\tau)}$, which is a correlation function for the outgoing signal. The result is given by:

$$\overline{w(t)\, w(t+\tau)} = \frac{a^2}{1-p^2} \int_0^\infty dt_1 \int_0^\infty dt_2 \; \overline{u(t-t_1)\; u(t+\tau-t_2)} \; .$$

$$\cdot \; e^{-p\omega(t_1+t_2)} \cos(\omega t_1 \sqrt{1-p^2}+\epsilon) \cos(\omega t_2 \sqrt{1-p^2}+\epsilon) ,$$

which can be transformed by introducing $\sigma = t_2 + t_1$ and $\delta = t_2 - t_1$ as new variables. The following expression is obtained:

$$\overline{w(t)\, w(t+\tau)} = \frac{a^2}{4p\omega\sqrt{1-p^2}} \int_0^\infty d\delta \left\{ S_2(\tau+\delta) + S_2(\tau-\delta) \right\} e^{-p\omega\delta} \cos(\omega\delta\sqrt{1-p^2}+\epsilon),$$

where $S_2(\tau) = \overline{u(t)\, u(t+\tau)}$, that is the Eulerian time correlation for $u(t)$ at a fixed point of space, as considered before in Section 11. In this way the correlation function for the outgoing signal w has been expressed by means of a correlation function for the incoming signal.

27. We can make the following use of this result. We first take $\tau = 0$, so that on the left-hand side we obtain the mean square value of w, which can be measured directly with the aid of a thermo-cross. If p is so small that $\cos\epsilon = \sqrt{1-p^2} \cong 1$, we find:

$$\overline{w^2} = \frac{a^2}{2p\omega} \int_0^\infty d\delta\; S_2(\delta)\, e^{-p\omega\delta} \cos(\omega\delta + \epsilon) .$$

The integral appearing here is related to integrals considered before. In Section 18 we wrote:

$$S_2(\tau) = \int_0^\infty d\omega \;\; \Gamma^*(\omega) \cos\omega\tau,$$

the inversion of which is:

$$\Gamma^*(\omega) = \frac{2}{\pi} \int_0^\infty d\tau\; S_2(\tau) \cos\omega\tau .$$

It was stated that $\Gamma^*(\omega)$ gives the energy spectrum with reference to frequencies in time. We now write:

$$\Gamma_I(\omega) = \int_0^\infty d\tau\; S_2(\tau)\; e^{-p\omega\tau} \cos\omega\tau ;$$

$$\Gamma_{II}(\omega) = \int_0^\infty d\tau\; S_2(\tau)\; e^{-p\omega\tau} \sin\omega\tau .$$

For $p \ll 1$, Γ_I will differ only slightly from Γ, so long as ω is not so large that $p\omega\tau$ will become comparable to 1 in the range where $S_2(\tau)$ has not yet dropped to zero. Usually one may expect that Γ_{II} will be smaller than Γ_I. If one now arranges the circuit so that a is proportional to $\sqrt{\omega p}$, we find:

$$\overline{w^2} \cong (\text{const. factor}) \cdot \left[\Gamma_I(\omega) - p \ \Gamma_{II}(\omega) \right] \quad .$$

In this way we see that, provided the filter is sufficiently selective and its scale factor is appropriately regulated, the mean square amplitude of the outgoing signal is nearly proportional to the spectral intensity $\Gamma^*(\omega)$.

The signal transmitted by the filter is approximately a harmonic function of the time with frequency ω. This is seen if we consider the correlation $\overline{w(t)\ w(t+\tau)}$ for a large value of τ, conveniently chosen so that $S_2(\tau+\delta) \cong 0$ for all $\delta > 0$. We then obtain:

$$\overline{w(t)\ w(t+\tau)} \cong \frac{a^2}{2p\omega} \cdot e^{-p\omega\tau} \left[\Gamma_{III}(\omega) \cos(\omega\tau + \epsilon) - \Gamma_{IV}(\omega) \sin(\omega\tau + \epsilon) \right]$$

where:

$$\Gamma_{III}(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau \ S_2(\tau) \ e^{-p\omega\tau} \cos \omega\tau \ ;$$

$$\Gamma_{IV}(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau \ S_2(\tau) \ e^{-p\omega\tau} \sin \omega\tau \ .$$

For sufficiently small p we may usually expect that $\Gamma_{III} \cong \Gamma$, while Γ_{IV} will be small of order p.

We thus see that w shows a nearly periodic correlation with frequency ω, gradually damped out through the factor $e^{-p\omega\tau}$.

Since we did not make any supposition about the character of the signal $u(t)$, apart from assuming that its correlation function $S_2(\tau)$ will become zero for τ exceeding a certain limit (this is necessary in order to make the integrals for Γ_{III} and Γ_{IV} convergent), we may say that the filter produced an almost periodic signal $w(t)$, with frequency ω determined by the properties of the circuit.

These considerations may give another proof of the fundamental nature of the correlation function. They show, moreover, that the problem of whether the incoming signal $u(t)$ should be considered as a superposition of an infinite number of harmonic components or as an arbitrary irregular function, is rather irrelevant.

CHAPTER V

Problems of Turbulent Spreading of Particles of Heat, and of Momentum

28. Spreading of Particles. When "particles" were considered in the preceding chapters, we assumed them to have the same density as the fluid and to be so small that their motion would give a satisfactory picture of the motion of elements of volume of the fluid. Spreading due to random motions could then be described by introducing the Lagrangian correlation for the velocity of these elements.

We now pass to particles with a density different from that of the fluid. Two phenomena present themselves. In the case of a field extending in the vertical direction, the particles will obtain a proper motion through gravity, either falling when their density exceeds that of the fluid, or rising in the opposite case. Second, the particles will not exactly follow the motion of the elements of volume in which they find themselves. A relative velocity appears and it is necessary to consider the forces which the particles experience from the surrounding fluid, and conversely, the reaction of the particles on the fluid.

The force experienced by a particle depends on: the difference in velocity between the particle and the surrounding fluid, its size, its shape, its position with respect to the vector of the relative velocity, and the density and viscosity of the fluid. Moreover, when the motion is variable in time, the accelerations, both of the fluid and of the particle, enter into the formulas, and an exact description may require data referring to the previous history of the motion.

It is only in the case of a spherical particle and for Reynolds numbers small in comparison with unity, that an equation has been given expressing the resistance as a function of these variables.*

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*Basset, Boussinesq, Oseen . . . . .
Tchen, C. M. "Mean Value and Correlation Problems Connected with the Motion of Small Particles Suspended in a Turbulent Fluid", Thesis Delft, 1947, Chapter 4. Medeleling No. 51 Uit Het Laboratorium Voor Aero-En Hydrodynamica der Technische Hogeschool Te Delft.

This equation can be integrated if the motion of the fluid surrounding the particle is known as a function of time.

Tchen arrived at the following results: (1) if the Reynolds number is small compared with unity, a constant velocity, either of fall or of rise, due to gravity acting on the difference in density, can be separated from the rest of the motion; (2) if we eliminate the motion produced by gravity by deducting its constant velocity from the actual velocity of the particle so that there remains only the random part of this velocity, it is found that the integral of the correlation function for the random velocity component w(t) in a given direction

$$D = \int_0^\infty d\tau \; \overline{w(t) \; w(t+\tau)}$$

is equal to the integral of the correlation function for the velocity component (in the same direction) of the random motion of the fluid surrounding the particle. Since the integral of the correlation function, according to the equations of Section 10, determines the rate of spreading of a cloud of particles, this result links the spreading of the particles to correlations existing in the motion of the fluid.

The correlation function for the motion of the fluid surrounding the particle, however, is not the same as the Lagrangian correlation for the motion of an element of volume of the fluid, since, as has been observed, a particle does not exactly follow the motion of an element of volume. In general, the particle will lag behind the motion of the element and its mean square velocity will be smaller than that of the element. If the element is of large size, compared with the wanderings of the particle relative to the center of the element, the particle on the whole may remain within the element; in that case the correlation for the fluid surrounding the particle will be practically identical with the Lagrangian correlation for the particle. However, if the particle often passes out of the element and penetrates into a neighboring one having a different velocity, we must expect that the correlation for the motion of the fluid surrounding the particle will be smaller than the Lagrangian correlation for a single element.

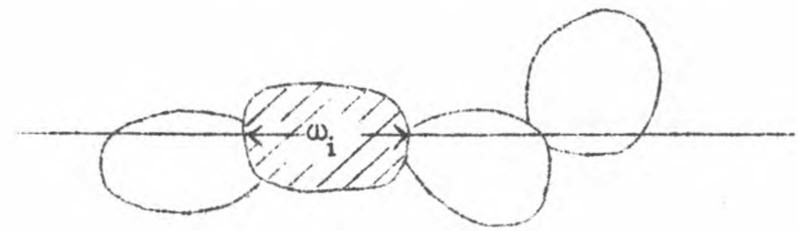
29. There exists at present no method for calculating exactly the effect of the passage of a particle from one element of volume into another on the correlation function for the particle. The picture which we have used, of more or less well defined elements of volume of the fluid, each having a movement of its own, is already an approximation. Although it helps to visualize what is happening, we must not forget that elements of volume change form and that at their boundaries there are transition regions between the motions of adjacent elements. Hence, superposed on what we have considered as "the motion of the element" there are distrubing small-scale motions; and if a particle comes near to a transition region, it may be caught by these small-scale motions. There are, consequently, a number of effects difficult to define, which can bring about the passage of a particle from a given element of volume into a neighboring one. In view of their random character, to a certain extent these effects can be described as a process of diffusion. In the case of very small particles, diffusion produced by molecular motions (as described by the classical physical concept of diffusion) must likewise be taken into account.

In view of this situation a different method of treatment has been developed which is applicable when we consider not only the motions of a few selected particles, but the behavior of a large number of particles and wish to investigate the effect of random turbulent movements on their spreading.

We again make use of the picture of more or less individualized elements of volume of the fluid which are constantly being shuffled about as a consequence of the turbulence. We suppose that the fluid contains a large number of particles of identical nature, and denote by n the number of particles per unit of volume of the fluid (actually, volume of the medium consisting of the fluid plus the particles contained in it). This number may be different for the various elements of volume, and for every element it can be a function of the time.

If the particles are all the same and are sufficiently small so that we may use linear equations for their motion, they will obtain a constant velocity of fall (or rise) under the influence of gravity, to be denoted by V_s. According to Tchen's result quoted before, this constant motion can be separated from the random motions. If the particles are of different sizes and shapes, the various types may have different velocities of fall and must be treated separately. We will deal, however, only with cases where the particles are uniform.

We consider a horizontal plane PP, at a given level z, in the field. At a certain instant t, this plane will cut through a number of elements of volume. Let us indicate the areas of the intersections by ω_i. If we know the particle density n_i, for each element of volume, the mean particle density over the horizontal plane PP

P  P

will be given by the sum $\Sigma\omega_i n_i$, taken over unit area of PP. This can be written as $\overline{n}$, where the bar over n denotes the mean value over the plane PP. Since gravity gives a vertical velocity of fall V_s to every particle, there will be a current of particles downward through the plane PP, of intensity $V_s\,\overline{n}$ per unit area and in unit time.

Superposed on this regular downward motion is the effect of the random motions. If the elements at the instant considered have vertical velocities w_i (positive if directed upward), there will be an aggregate upward transport of particles amounting to

$$\Sigma\omega_i\, n_i\, w_i = \overline{n\,w}$$

per unit area and in unit time. We must find out how this transport through random motions can be related to quantities characterizing the average distribution of the particles over the field and the intensity of the turbulence.

30. For this purpose it is necessary to give attention to the <u>exchange of particles between adjacent elements of volume</u>. This exchange will influence the value of n for an element, if the average particle density in the surrounding elements is different. Let us denote this average "surrounding particle density" by n^*. The rate of change of n due to exchange of

particles, with a certain degree of approximation, can be put proportional to $n^* - n$, as follows:

$$\frac{dn}{dt} = \lambda_1(n^* - n) .$$

Here λ_1 is a coefficient depending on a number of unknown factors, among which are the dimensions of the element considered and the intensity of the small-scale motions (eventually including classical diffusion due to molecular effects). The factor λ_1 may be different for elements of different size, but if we classify the elements we may suppose that λ_1 will have approximately the same value for elements of equal size. A general average can then be taken at a later stage of the calculations.

We had already introduced the average value $\bar{n}$ of the particle density over the plane PP, which the element we are considering is just crossing. This general average value need not be the same as n^*. The two quantities, however, cannot differ much. The average over the plane PP includes a contribution from the element under consideration (with density n), a large contribution from its surroundings (with density n^*) and a contribution from further elements. It seems possible under these circumstances to write:

$$\bar{n} = an^* + (1 - a)\, n,$$

where a is a coefficient probably not differing much from unity. We can then transform the expression for dn/dt into:

$$\frac{dn}{dt} = \lambda(\bar{n} - n) , \tag{1}$$

where $\lambda = \lambda_1/a$ is a new coefficient, which henceforth will be used instead of λ_1. It can be roughly assumed that λ is inversely proportional to the square of the diameter of the element.

Equation (1) can be used to find how the particle density n in the element under consideration has appeared as the result of exchanges with neighboring elements during its past history. The present value of n can be expressed by the integral of (1):

$$n = \lambda \int_0^{\infty} dt' \, e^{-\lambda t'} \, \bar{n}(t - t') \tag{2}$$

Here $\overline{n}(t - t')$ indicates the mean value of the particle density at the level where the element found itself at the instant $t - t'$. To find this particle density, we assume that the average particle density is stationary (independent of the time), and that, in the neighborhood of the plane PP, it can be considered as a linear function of z, say,

$$\overline{n} = C + bz.$$

The level z where the center of the element finds itself at the instant $t - t'$ can be found by means of an integration of the vertical velocity w of the element:

$$z = zp - \int_0^{t'} w(t - t'')\, dt''$$

It follows that the value of $\overline{n}$ for the level at which the element found itself at the instant $t - t'$, is given by:

$$(3) \quad \overline{n}(t - t') = (C + bz_P) - b \int_0^{t'} w(t - t'')\, dt'' .$$

31. When the expression (3) is substituted into (2) we obtain:

$$n = (C + bz_P) - b\lambda \int_0^{\infty} dt'\, e^{-\lambda t'} \int_0^{t'} w(t - t'')\, dt''$$

$$= (C + bz_P) - b \int_0^{\infty} dt'\, e^{-\lambda t'}\, w(t - t') .$$

The first term on the right-hand side is nothing else than the average particle density $\overline{n}$ at the level of PP where the element finds itself at the instant t; it is independent of the particular element of volume under consideration. We may therefore write:

$$(4) \quad n - \overline{n} = -\frac{d\overline{n}}{dz} \int_0^{\infty} dt'\, e^{-\lambda t'}\, w(t - t'),$$

where b has been replaced by $d\overline{n}/dz$.

We must introduce this expression into the formula $\overline{n.w}$ for the transport due to random motions. Since in a field of homogeneous turbulence the assumption of random motions implies that there is no average flow, the mean value of w itself over the plane PP must be zero (incompressibility being presupposed). Hence, in determining $\overline{nw}$, the term $\bar{n}$ in (4) drops out and there remains:

$$\text{(5)} \qquad \overline{nw} = -\frac{d\bar{n}}{dz}\int_0^\infty dt' \, e^{-\lambda t'} \, \overline{w(t-t')\,w(t)} \; .$$

For every value chosen for t' the mean value $\overline{w(t-t')\,w(t)}$ occurring under the integral sign is taken over the plane PP; this means it refers to all the elements of volume crossing the plane PP at the instant t. However, in the case of stationary turbulent motion, this mean value cannot differ from the mean value of $\overline{w(t-t')\,w(t)}$ calculated for the history of a single element of volume; that is, calculated as a time-mean value by giving a series of values to t, keeping t' constant. Hence this mean value is equal to the Lagrangian correlation for the motion of an element of volume; consequently we shall write R_w for it. In this way the expression for the transportation of particles through the plane PP per unit area and in unit time can be written:

$$\text{(5a)} \qquad \overline{nw} = -\frac{d\bar{n}}{dz}\int_0^\infty dt' \, e^{-\lambda t'} \, R_w(t') \; .$$

This can be brought into the form:

$$\text{(5b)} \qquad \overline{nw} = -D_p \frac{d\bar{n}}{dz} \; ,$$

where, written somewhat more accurately:

$$\text{(5c)} \qquad D_p = \int_0^\infty dt' \, e^{-\lambda t'} \, \overline{w(t-t')\,w(t)} \; .$$

This quantity D_p is the "<u>turbulent diffusion coefficient</u>". In defining the average value, we now have averaged also over various values of the exponential factor, since elements of volume can have different sizes and will have different exchange coefficients λ. It will be evident that when the rate of exchange of particles between neighboring elements on the whole is slow, so that all λ are small, D_p will practically be given by the ordinary integral of the Lagrangian correlation function, as considered before in Section 10. When the rate of exchange is large, the turbulent diffusion coefficient will be smaller provided the correlation curve is of simple type (we come back to this point in Section 37). These results confirm those deduced from the general reasoning of Section 28.

The expression (5c) for D_p is also applicable when, instead of particles carried by the fluid, we consider a dissolved substance. Instead of the particle density n, we then better consider the concentration c (defined as mass per unit volume) of the dissolved substance. In this case probably λ will mainly depend on true molecular diffusion.

32. The complete expression for the strength of the current of particles, combining the transport due to gravity with that due to the random motions, is:

$$(6) \qquad M = -V_s \bar{n} - D_p \frac{d\bar{n}}{dz} .$$

In the case of a stationary field M must have a constant value (independent of z), which will be zero if the boundaries of the field cannot be penetrated by the particles.

When the value is not constant, the field cannot be stationary. The deduction given above loses somewhat of its applicability, but for slow rates of change in a field of homogeneous turbulence, we can use the equation:

$$(7) \qquad \frac{\partial \bar{n}}{\partial t} = -\frac{\partial M}{\partial z} = V_s \frac{\partial \bar{n}}{\partial z} + D_p \frac{\partial^2 \bar{n}}{\partial z^2} .$$

If the turbulence is not homogeneous, the definition of the correlation function $\overline{w(t-t')\,w(t)}$ will require adjustment and may differ from the Lagrangian correlation. Its value will become a function of the level z for which M must be obtained. Equation (7) is replaced by:

$$(8) \qquad \frac{\partial \bar{n}}{\partial t} = V_s \frac{\partial \bar{n}}{\partial z} + \frac{\partial}{\partial z}\left(D_p \frac{\partial \bar{n}}{\partial z}\right).$$

33. Reaction of the Particles on the Motion of the Fluid. We have assumed in the preceding calculations that gravity gives a constant velocity of fall V_s to the particles. This means that the extra weight of particles, above that of the fluid displaced by them, is balanced by the average resistance they experience from the fluid. In turn they exercise a downward force on the fluid equal to the excess of weight.

We can make use of this result in writing down an equation of motion for an element of volume of the fluid. Instead of referring to the complete hydrodynamical equations, we shall use the following simplified form:

$$\rho \frac{dw}{dt} = -\frac{d\overline{p}}{dz} + F_z - g\rho - K\rho w. \tag{9}$$

On the right-hand side the first term represents the mean pressure gradient, which is connected with the mean density by the relation: $-d\overline{p}/dz = g\overline{\rho}$. The term F_z has been written for the effect of randomly changing pressures around the element connected with the turbulent motion of the field. (This term may include random effects of friction.) Then comes the action of gravity on the element we are considering, $-g\rho$. Finally the term $-K\rho w$ has been introduced as a measure for the average resistance experienced by the element in its motion between the surrounding elements. This expression is no more than an approximation.

The density ρ of an element depends on the number of particles per unit volume contained in it, or on the mass concentration of suspended or dissolved material. We assume that there is a linear relation between concentration and density of the type:

$$\rho - \rho_o = \alpha n,$$

where ρ_o represents the density of the pure fluid without particles or dissolved material, while α is a constant factor. It then follows that:

$$\overline{\rho} - \rho_o = \alpha\overline{n}; \quad \rho - \overline{\rho} = \alpha(n - \overline{n}); \quad \frac{d\overline{\rho}}{dz} = \alpha\frac{d\overline{n}}{dz}.$$

Our previous equation (4) consequently can be replaced by:

$$\rho - \overline{\rho} = -\frac{d\overline{\rho}}{dz}\int_o^\infty dt'\, e^{-\lambda t'}\, w(t-t'). \tag{4a}$$

This can be used to eliminate the group of terms: $-d\overline{p}/dz - g\rho = -g(\rho - \overline{\rho})$.

from Eq. (9). Since we then have taken sufficient account of the density fluctuations, we replace ρ by $\bar{\rho}$ in the terms $\rho(dw/dt)$ and $-K\rho w$.* In this way we arrive at an integro-differential equation for w. By means of a simple transformation it can be reduced to the following ordinary differential equation of the second order:

$$(10) \qquad \frac{d^2w}{dt^2} + (K+\lambda)\frac{dw}{dt} + (\beta g + K\lambda)\,w = \frac{1}{\bar{\rho}}\left(\frac{dF_z}{dt} + \lambda F_z\right).$$

Here the letter β has been used for

$$(10a) \quad \ldots\ldots \quad \beta = -\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dz} = -\frac{d}{dz}(\ln\bar{\rho}).$$

<u>Appendix to Section 33</u>: Derivation of Eq. (10)

When (4a) is substituted into (9), we obtain

$$\rho\frac{dw}{dt} = g\frac{d\rho}{dz}\int_0^\infty dt'\, e^{-\lambda t'}\, w(t-t') + F_z - k\bar{\rho}w,$$

which can also be written:

$$\frac{dw}{dt} + kw = -\beta g\int_0^\infty dt'\, e^{-\lambda t'}\, w(t-t') + \frac{F_z}{\bar{\rho}}$$

Taking the derivative with respect to the time:

$$\frac{d^2w}{dt^2} + k\frac{dw}{dt} = \beta g\int_0^\infty dt'\, e^{-\lambda t'}\,\frac{\partial w}{\partial t} + \frac{1}{\bar{\rho}}\frac{dF_z}{dt}$$

The integral on the right-hand side by means of partial integration can be transformed into:

$$-\beta g\, w(t) + \beta g\lambda\int_0^\infty dt'\, e^{-\lambda t'}\, w(t-t').$$

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*In a fluid <u>heavily</u> loaded with sediment (or with a dissolved substance), the increase in <u>density</u> can materially affect the inertia of the elements. The simplification introduced in the text then is not allowed.

The latter integral can be eliminated with the aid of the undifferentiated equation, giving the result:

$$\frac{d^2w}{dt^2} + k\frac{dw}{dt} = -\beta gw - \lambda\left(\frac{dw}{dt} + kw - \frac{F_z}{\bar{\rho}}\right) + \frac{1}{\bar{\rho}}\frac{dF_z}{dt} .$$

This can easily be rearranged so as to obtain the form (10).

34* We first discard the effect of the forces F_z and consider the homogeneous equation:

$$\text{(10b)} \qquad \frac{d^2w}{dt^2} + (K+\lambda)\frac{dw}{dt} + (\beta g + K\lambda)\,w = 0.$$

If the density gradient, $d\bar{\rho}/dz = 0$, $\beta = 0$, and the solutions of this equation will represent damped motions. A positive value of $d\bar{\rho}/dz$ makes β negative. When $\beta g + K\lambda < 0$, solutions appear which represent motions increasing in time. This means that a situation in which the density increases upwards (heavier layers on top of lighter layers) will be unstable if the gradient exceeds a certain limit. With a negative density gradient (lighter layers above the heavier ones) all solutions of (10b) are damped; hence situations with negative $d\bar{\rho}/dz$ are always stable. In the case

$$\beta g > \frac{(K-\lambda)^2}{4}$$

the solutions will have a periodic character.

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*In preparing this section and section 36 - 39, great help has been derived from a report by ir. J. C. Schönfeld of the "Rykswaterstaat" (Government Water Board) in Holland, written for a seminar on turbulence at the Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool, held in the spring of 1950.

When the right-hand side is re-introduced, the appropriate solution of (10) for the last mentioned case can be written in the form:

$$(11) \qquad w = \frac{1}{n}\int_0^{\infty} dt' \, e^{-kt'} \sin nt' \ \varphi(t - t') ,$$

where $k = \frac{1}{2}(K + \lambda)$; $n^2 = \beta g - \frac{(K - \lambda)^2}{4}$; while φ has been used as an abbreviation for

$$\varphi = \frac{1}{\overline{\rho}}\left(\frac{dF_z}{dt} + \lambda F_z\right) .$$

Much now depends on the magnitude of K and λ. If the resistance, expressed by the coefficient K, would be due mainly to ordinary viscosity, this coefficient appears to be of the order of magnitude $3\nu/R^2$, ν being the kinematic viscosity of the fluid and R a mean radius of the element. If the exchange of the transported material is exclusively due to ordinary molecular diffusion, it is generally found that λ is much smaller than K. When effects of small scale motions must be taken into account, the value is difficult to estimate; but it will be seen already how much depends on the size of the elements of volume which can be considered as moving more or less individually. With large elements, complications present themselves which have not appeared in the preceding deductions; for instance, exchange of particles or of dissolved substance will mainly be limited to surface layers; further, in determining the transport of material through the plane PP, considered before, we must give attention to whether the center of the element is in this plane or is at an appreciable distance above or below it.

If there is no pre-existing turbulence, but only a stratified system in which the forces indicated by F_z would have the character of minor disturbing effects, a positive value of $d\overline{\rho}/dz$, will always bring instability. The dimensions of the elements of volume may depend on features of the distribution of the disturbances F_2 and may become large. In the case of a negative $d\overline{\rho}/dz$, all motions will be damped, but there may appear motions of periodic character with very small damping.

If turbulence already exists, the dimensions of the randomly moving elements of volume will be mainly determined by the character of the turbulence. The transport phenomena can then be considered as secondary effects. Nevertheless, the circumstance that Eq. (10b) leads to damped motions when $d\bar{\rho}/dz < 0$ indicates that loss of energy is connected with the transport phenomena. This can also be seen from Eq. (9) if this equation is multiplied by w and a mean value is taken over all elements crossing the plane PP; we then obtain

$$(12) \qquad \frac{d}{dt}\left(\frac{1}{2}\overline{\rho w^2}\right) = \overline{F_z w} - g\overline{\rho w} - K\overline{\rho w^2} .$$

The left-hand side describes the increase of the average kinetic energy of the vertical motion. The mean value $-\overline{w}(d\bar{\rho}/dz)$, in which $d\bar{\rho}/dz$ is a constant independent of w, drops out. The term $\overline{F_z w}$ represents the average expenditure of energy by the fluctuating pressure gradients maintaining the vertical random motion. This energy must be derived from other forms of motion, a question to which we shall give some attention later on, in connection with the problem of exchange of momentum. At the present moment this quantity may be considered as given. The last term of the equation, $K\overline{\rho w^2}$, represents a loss of energy through friction; this is part of the ordinary dissipation always to be found in turbulent motion. Finally, the term $-g\overline{\rho w}$ is the only one in which we take into account the correlation between ρ and w. According to (4) and (5) it can be written:

$$(12a) \qquad -g\overline{\rho w} = g\frac{d\bar{\rho}}{dz}\int_0^\infty dt'\,\overline{e^{-\lambda t'}\,w(t-t')\,w(t)} = -\beta g\bar{\rho}D_p .$$

For negative $d\bar{\rho}/dz$, so that $\beta > 0$, this term gives the extra loss of energy connected with the turbulent spreading of a heavy admixture.

When the field is stationary and when the current M, as given by Eq.(6) is zero, we have:

$$-D_p\frac{d\bar{n}}{dz} = V_s\bar{n} ,$$

which can be transformed into:

$$- D_p \frac{d\bar{\rho}}{dz} = V_s (\bar{\rho} - \rho_o) .$$

This makes it possible to write the expression for the loss of energy also in the following form :

$$(12b) \qquad - g\, \overline{\rho w} = - g(\bar{\rho} - \rho_o)\, V_s ,$$

which shows that the loss of energy is given by the excess of weight per unit volume multiplied by the velocity of fall. It is thus equal to the work which must be done against gravity to keep the particles in suspension and to counteract their sedimentation.

Since, in particular, the larger elements of volume have the greatest density differences with their surroundings (we have seen that diffusion operates most rapidly towards equalization when the dimensions of the element are small), we may expect that the motion of the larger elements will suffer most from this loss of energy. Hence the presence of particles kept in suspension will have the tendency to reduce the large scale turbulent motions.

It is possible that a large negative value of $d\bar{\rho}/dz$ may favor the appearance of a more or less periodic motion of elements of sufficient size. If such motions appear, they may influence the form of the correlation curve. We come back to this in Section 37.

35. Transport of Heat. The transport of heat depends on turbulence in much the same way as the transport of particles, etc., but certain details require separate consideration.

In the first place, temperature changes produce changes in density and thus affect the volume of the randomly moving elements. Although the change of volume is small, account of it must be taken in calculating the amount of work done in transporting elements of volume in a field with a temperature gradient. We, therefore, introduce the density ρ of the element and obtain an aggregate transport of mass through our plane PP: $\Sigma\omega_i\, \rho_i\, w_i = \overline{\rho w}$. In the present case, although the elements of volume may exchange heat and thus change their density, we can assume that there is no resulting transport of matter. Consequently the condition $\overline{w} = 0$, which we could use for incompressible motion, must now be replaced by $\overline{\rho w} = 0$.

If the temperature of an element of volume is T, the transport of heat, per unit area and in unit time, is given by:

$$\Sigma \omega_i \rho_i w_i C_v T_i = C_v \overline{\rho w T},$$

C_v being the specific heat at constant volume. At the same time, work is done by the pressure to the amount:

$$\Sigma p_i \omega_i w_i = R \Sigma \omega_i \rho_i w_i T_i = R \overline{\rho w T};$$

R here being the gas constant, so that $p = R\rho T$. Hence the total transport of energy is given by:

$$Q = C_p \overline{\rho w T}, \tag{13}$$

where $C_p = C_v + R$ is the specific heat at constant pressure.

Further, the temperature of an element of volume duringits random movement does not only change through conduction but also because of different pressure. There will be a systematic effect connected with the mean pressure gradient in the field, which itself is connected with gravity. Applying Poisson's law, according to which the absolute temperature is proportional to pressure $(\gamma - 1)/\gamma$ for adiabatic changes of state, we have:

$$dT = \frac{\gamma - 1}{\gamma} \frac{T}{p} dp$$

and we find a rate of change of temperature for an element of volume possessing the vertical velocity v of amount:

$$\left(\frac{dT}{dt}\right)_{Poisson} = \frac{\gamma - 1}{\gamma} \frac{\overline{T}}{\overline{p}} w \frac{dp}{dz} = - \frac{\gamma - 1}{\gamma} \frac{\overline{\rho}T}{\overline{p}} g w .$$

Here $\gamma = C_p/C_v$, where C_p is the specific heat at constant pressure, C_v is specific heat at constant volume. Making use of the equation of state for an ideal gas, $p = R \rho T$, we have $C_p - C_v = R$, if consistent units are used, i.e., specific heat measured in units of energy per degree, etc. We now can write:

$$\Gamma = \frac{\gamma - 1}{\gamma} \frac{g}{R} = \frac{g}{C_p}$$

This quantity is called the "adiabatic lapse rate" of the temperature. It should be noted that if we had investigated heat transport in a horizontal direction, in a field which does not show a mean pressure gradient in the direction we are considering, Γ must be replaced by zero. Adding $(dT/dt)_{Poisson}$ to the rate of change by conduction of heat from an element to its surroundings, or inversely, for which we again will make use of a coefficient λ , we obtain:

$$(14) \qquad \frac{dT}{dt} = \lambda(\overline{T} - T) - \Gamma w + \psi \quad .$$

An extra term ψ has been introduced, which can be used if there are other effects influencing the temperature, for instance radiation or condensation phenomena. When there is no need to take such effects into account, ψ can be omitted.

If we use the letter b now to denote the mean temperature gradient $\overline{dT}/dz$, we can write:

$$\overline{T}(t - t') = (C + b\, z_P) - b \int_0^{t'} w(t - t'')\, dt''$$

for the mean temperature at the level where our element of volume found itself at the instant $t - t'$. With the aid of this formula, the integral of (14) can be brought into the form:

$$(15)\ T - \overline{T} = -(b + \Gamma) \int_0^{\infty} dt'\, e^{-\lambda t'}\, w(t - t') + \int_0^{\infty} d\lambda'\, e^{-\lambda t'}\, \psi\,(t - t')\ ,$$

$\overline{T}$ here denoting the mean temperature at the level where the element finds itself at the instant t, i.e., at the plane PP. The transport of energy across this plane, per unit area and in unit time, now becomes:

$$(16)\ Q = -(b + \Gamma) C_p \overline{\rho} \int_0^{\infty} dt'\, \overline{e^{-\lambda t'} w(t-t')\, w(t)} + C_p \overline{\rho} \int_0^{\infty} dt'\, \overline{e^{-\lambda t'}\, \psi(t-t')\, w(t)}\ .$$

In the expressions for the averages we have taken apart the mean density $\bar{\rho}$ since the inaccuracy introduced in this way can be neglected.

If we leave aside the effects included under the term ψ, it will be seen that the heat transport in the vertical direction depends on the factor $b + \Gamma$. It vanishes when the mean temperature gradient $b = d\bar{T}/dz = - \Gamma$, that is, when the temperature decreases according to the adiabatic lapse rate. The so-called potential temperature then is constant in the vertical direction.

An interesting feature may be connected with the term ψ. Since the temperature of the element influences its density and thus its buoyancy with respect to the surrounding elements, ψ can have influence on the motion and a correlation between ψ and w is possible. This can have the effect that $b + \Gamma$ sometimes must have a small positive value to make Q vanish.

36. Transfer of Momentum in a Stratified Turbulent Flow. Thus far the random motions we have been considering were in the vertical direction. Reference to turbulent motions in other directions has been made only insofar as the pressure force F_z in Eq. (9) might depend on them.

If we consider a stratified turbulent flow in which the mean velocity of flow is a function of z, the vertical motion of the elements of volume will bring about a transfer of horizontal momentum. In principle, similar relations are effective in this phenomenon as have been considered in the preceding sections. The quantity transported is the horizontal momentum component (in the direction of the mean flow). We, therefore, need an equation describing how the horizontal velocity is influenced by the motion of the surrounding elements. For this we use:

$$\bar{\rho}\frac{du}{dt} = F_x - K\bar{\rho}(u - \bar{u}) \quad (17)$$

In this equation, where attention is not directed to the transportation of foreign material or of heat, the mean density can be used. The term F_x, which is similar to F_z occurring in Eq. (9), represents the effect of pressures giving a resultant in the horizontal direction, while K again has been used to describe the resistance experienced by the horizontal motion. The mean velocity of flow in the horizontal direction $\bar{u}$ (often written U) is a function of z.

If we write $d\overline{u}/dz = U'$, the integral of (17) becomes:

$$u - \overline{u} = - U' \int_0^\infty dt' \; e^{-Kt'} \; w(t - t') + \frac{1}{\overline{\rho}} \int_0^\infty dt' \; e^{-Kt'} \; F_x(t - t) .$$

The transfer of momentum is now given by:

$$(18) \quad \overline{\rho} \; \overline{uw} = - \overline{\rho} U' \int_0^\infty dt' \; \overline{e^{-Kt'} w(t - t') \, w(t)} + \int_0^\infty dt' \; \overline{e^{-Kt'} F_x(t - t') \, w(t)} .$$

It is generally assumed that there is no correlation between w and F_x and the last term can be left out.

The transfer of momentum can be described as a <u>shearing stress</u> acting on the mean field of flow. The stress is ordinarily defined with the opposite sign, so that:

$$(18a) \quad \tau_{xz} = - \overline{\rho} \; \overline{uw} = \overline{\rho} \, U' \int_0^\infty dt' \; \overline{e^{-Kt'} w(t - t') \, w(t)} \; .$$

37. We have now obtained three transfer coefficients:

$$(I) \qquad D_p = \int_0^\infty dt' \; \overline{e^{-\lambda t'} \; w(t - t') \; w(t)}$$

compare Eq. (5c) for particles, with λ referring to particle exchange between neighboring elements;

$$(II) \;.. \; \overline{\rho} \, C_p D_q = C_p \overline{\rho} \int_0^\infty dt' \; \overline{e^{-\lambda t'} \; w(t - t') \, w(t)}$$

compare Eq. (16) for heat, with λ referring to heat conduction between neighboring elements;

$$(III) \;.. \; \rho \; D_m = \overline{\rho} \int_0^\infty dt' \; \overline{e^{-Kt'} \; w(t - t') \; w(t)}$$

compare Eq. (18) for momentum K referring to the resistance experienced by an element in its horizontal motion. All expressions depend on the same correlation in the vertical motion of the elements, but the exponential factors are different. If the exponential factor is omitted altogether, we arrive at

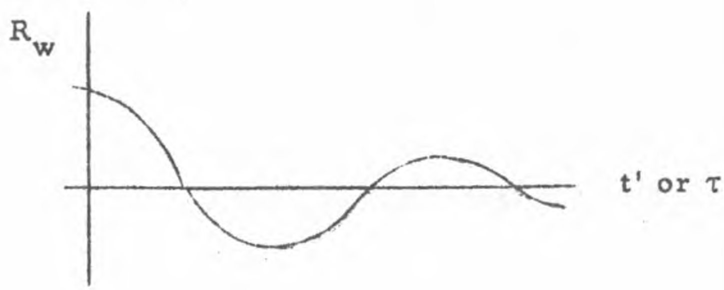
$$\text{(IV)} \quad \ldots \quad D = \int_0^{\infty} dt' \; \overline{w(t - t') \; w(t)} \, ,$$

which is equal to the coefficient of turbulent spreading for particles completely following the motion of the elements of volume, as discussed in Section 10 (Chapter II).

Comparing D_m and D_p, if we can assume $K > \lambda$, as is the case when the exchange of material between elements of volume is due to molecular diffusion only, one can expect that in general

$$D_m < D_p \, ,$$

provided the correlation curve is of simple shape (R_w decreasing from its maximum to zero without change of sign). However, if, in the case of a large density gradient where there is an important number of elements of volume with a (damped) periodic motion, there seems to be a possibility for R_w to be of the form:



In such a case it is possible that

$$D_m > D_p$$

for particular values of K, even if $K > \lambda$.

Measurements by J. P. Jacobsen* in the Danish Waters have shown

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*Rapports et Proc. Verbaux des Reunions du Conseil Permanent pour l'Exploration de la Mer, vol. 64, p. 59, 1930.

that such cases are found in nature, although cases with $D_m < D_p$ are found more commonly.

The investigations made by V. A. Vanoni* likewise have given cases where D_p can be either larger or smaller than D_m. It was found that for fine material D_p tended to be larger than D_m, which is in accordance with the first case mentioned above, so that we are led to assume that with fine sediment $K > \lambda$. With coarser sediment the opposite relation was obtained. I would suppose that in this case the explanation must not be sought in the appearance of periodic motions, but rather in an increased tendency of coarse particles to escape from elements of volume to neighboring ones, which might make the factor λ for them to be much larger than the coefficient K.

38. When a shearing stress τ_{xz} acts on a field of flow with a mean velocity gradient $du/dz = U'$, the energy transmitted to the turbulent motions per unit volume and in unit time is:

$$U'\tau_{xz} = \bar{\rho} D_m (U')^2 \tag{19}$$

This energy is derived from the work done by the exterior forces driving the fluid, for instance in the case of flow through a tube from the longitudinal pressure drop, and in the case of an inclined canal from gravity. It is finally spent through friction in the turbulent motion. If there is a negative density gradient, work will also be spent as a consequence of turbulent mixing. According to (12a), this latter work is given by: $\beta g \bar{\rho} D_p$. Hence we may write as a general expression for the energy balance:

$$U'\tau_{xz} = \bar{\rho} D_m (U')^2 = \beta g \bar{\rho} D_p + \text{work lost through} \tag{20}$$

viscous friction. Since the work lost through viscous friction necessarily is positive, we must have:

$$D_m/D_p > r , \tag{21}$$

where r has been written for the dimensionless parameter:

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* "Transportation of Suspended Sediment in Water", Trans. Amer. Soc. Civil Engineers, <u>111</u>, p. 67-133, 1946.

(21a) $$r = \beta g/(U')^2$$

which was first introduced by L. F. Richardson.* If r is too large, so that this inequality cannot be satisfied, turbulence will be impossible. This again explains the stabilizing effect of large negative density gradients.

The inequality (21) has been given in that form by G. I. Taylor.** In former work it had been tacitly assumed that D_p and D_m would always be equal, so that the critical value of r would be unity. Measurements by J. P. Jacobsen proved that values of r much above unity are well compatible with turbulence. From simultaneous measurements of the velocity and the salt distribution, data could be obtained making possible the calculation of D_m and D_p, and which showed that on the whole the relation (21) is satisfied. Compare e.g., S. Goldstein.***

39. In order to bring Eq. (20) into a more complete form, we repeat the approximate equations of motion (17) and (9) and add to them an equation of similar type for the movements in the y-direction. The following system is then obtained:

(22) $$\begin{cases} \bar{\rho}\,\dfrac{du}{dt} = F_x - k\bar{\rho}\,(u-\bar{u}) \\[2ex] \bar{\rho}\,\dfrac{dv}{dt} = F_y - k\bar{\rho}\,v \\[2ex] \bar{\rho}\,\dfrac{dw}{dt} = F_z - k\bar{\rho}\,w - \dfrac{d\bar{p}}{dz} - g\rho \end{cases}$$

In those terms which do not depend on density differences, we have replaced ρ by $\bar{\rho}$. Although, strictly speaking, $\bar{\rho}$ is a function of z, no error of importance is made if we treat $\bar{\rho}$ as a constant factor in these terms. In the third equation $-d\bar{p}/dz - g\rho = -g(\rho - \bar{\rho})$, and here the difference between ρ and $\bar{\rho}$ remains important. We further write: $u - \bar{u} = u_1$, so that

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*Proc. Royal Soc., London, A, 97, p. 354, 1930.

**Rapp. et Proc. Verb. des Reunions du Conseil Permanent Intern. pour l'Exploration de la Mer., Vol. 76, p. 35, 1931.

***Recent Developments in Fluid Dynamics, Oxford, 1938, Vol. I, pp. 229-232.

u_1 represesents the difference between the velocity of an element of volume and the mean velocity of the fluid for the same value of z. For an observer following the element, $\bar{u}$ changes when the z-coordinate of the element changes, and for such an observer $d\bar{u}/dt = w \cdot d\bar{u}/dz = wU'$. Hence $du/dt = du_1/dt + wU'$ and we can transform the first equation of the system (22) into:

$$\text{(22a)} \qquad \bar{\rho}\,\frac{du_1}{dt} = F_x - k\,\bar{\rho}\,u_1 - \bar{\rho}\,wU' .$$

We multiply this equation by u_1 and add to it the second and third equations of (22), multiplied by v and w, respectively. This gives;

$$\frac{1}{2}\bar{\rho}\,\frac{d}{dt}(u_1^2 + v^2 + w^2) = (F_x u_1 + F_y v + F_z w) - k\bar{\rho}(u_1^2 + v^2 + w^2) -$$

$$- g(\rho - \bar{\rho})w - \bar{\rho}\,u_1\,wU' \,\ldots\ldots\ldots \quad \text{(A)}$$

To obtain the energy balance we calculate the mean values of both sides of this equation, following the element of volume through its motion. The left-hand member will become zero for a stationary field. In the right-hand member

$$- g\overline{(\rho - \bar{\rho})w} = - g\,\overline{\rho w} = - \beta g \bar{\rho} D_p ,$$

according to Eq. (12a), Section 34. To find the mean value of the last term of (A), we refer to the integral of Eq. (17), given in Section 36. It will be seen that the mean value of $\bar{\rho} u_1 w$ is the same as that of $\bar{\rho}$ uw and is equal to:

$$\bar{\rho}\;\overline{u_1 w} = - \bar{\rho}\,U' \int_0^\infty \overline{dt'\, e^{-kt'}\, w(t-t')\, w(t)}$$

(compare Eq. (18), and the remark following it). If we can make use of (18a) and also of (III), Section 37, we can write this:

$$\bar{\rho}\;\overline{u_1 w} = - \tau_{xz} = - \bar{\rho} D_m U' .$$

Hence the result of taking the mean value of Eq. (A) above is

$$0 = \overline{(F_x u_1} + \overline{F_y v} + \overline{F_z w)} - k\bar{\rho}(\overline{u_1^2} + \overline{v^2} + \overline{w^2}) - \beta\, g\bar{\rho} D_p + \bar{\rho} D_m (U')^2 .$$

We introduced the F_x, F_y, F_z as irregular pressure forces, depending on the turbulence; they produce a coupling between the various components and can transmit energy from one form of turbulence to other forms. It is logical, therefore, to suppose:

$$\overline{F_x u_1} + \overline{F_y v} + \overline{F_z w} = 0 ,$$

which means that for the turbulence as a whole there shall be no gain nor loss of energy through the action of these forces.

The energy balance for stationary turbulence consequently reduces to:

$$\bar{\rho}\, D_m (U')^2 = \beta g \bar{\rho}\, D_p + k \bar{\rho}\, (\overline{u^2} + \overline{v^2} + \overline{w^2}) . \tag{24}$$

This equation expresses that all energy derived from the main motion with its velocity gradient U' is spent, partly in viscous friction, partly in overcoming gravity insofar as foreign material is carried.

40. <u>Note on the Concept of Mixing Length</u>. The expressions for D_p, D_q and D_m, considered in Section 37, have the dimensions $(\text{velocity})^2 \cdot (\text{time})$, which is equivalent to (velocity) · (length). In many considerations on the processes of exchange of momentum and of mixing, they are replaced by the mean value of the product of the velocity component $w(t)$ the velocity of the element of volume when it crosses the level PP into a length ℓ, which is considered to represent the distance travelled by the element since the instant when it made itself free of its surroundings for the last time before coming to PP. This distance is often called the "free path", in reminiscence of a similar quantity occurring in the kinetic theory of gases. It is then assumed that the element brings with it the value of the mean horizontal velocity, or the mean concentration of matter, or the mean temperature as found at the level $z_p - \ell$.

Formally there is no objection to writing, e.g.:

$$\ell = \int_0^\infty dt' \, e^{-Kt'} w(t - t')$$

which makes it possible to write

$$D_m = \overline{\ell w} .$$

But this way of writing somewhat obscures the circumstance that the exponential function in the expressions for D_p, D_q and D_m will be different. The concept of an element of volume, freeing itself completely from the surroundings in which until that instant it had been caught, is more crude and is less adaptable than the idea of a gradual exchange. In mixing length theory very often the same length is used for all cases and, consequently, it is often assumed that the three transfer coefficients have the same value. This would make it possible, for instance, to make calculations on heat transfer when the magnitude of the momentum transfer could be obtained from considerations on the force equilibrium. Difficulties, however, have been encountered in interpreting certain observational results concerning simultaneous momentum and heat transfer. This has led to an investigation into the problem of whether in some cases, instead of transfer of momentum (that is, of velocity), one whould not rather consider transfer of vorticity. We come back to this point in Section 47.

CHAPTER VI

Features of the Navier-Stokes Equations

41. For a fluid of constant density, the Navier-Stokes equations of motion have the following form:

$$(1) \qquad \rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho\nu\,\Delta\, u_i$$

$$(2) \qquad \frac{\partial u_i}{\partial x_i} = 0$$

For convenience in notation the coordinates have been denoted by x_1, x_2, x_3 and the components of the velocity by u_1, u_2, u_3. Where repeated indices occur, it is understood that a summation is carried out.

By virtue of the equation of continuity, it is possible to write:

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i u_j) .$$

Equations (1) consequently can be transformed into:

$$(1a) \ldots . \quad \rho \frac{\partial u_i}{\partial t} = - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}(\rho u_i u_j) + \rho\nu \Delta u_i .$$

Sometimes the original form of the nonlinear terms is more convenient; in other cases the new form has advantages.

We will assume that the field is stationary in the statistical sense, so that time mean values will exist for any variable quantity. The mean values of the velocity components will be denoted by U_1, U_2, U_3 and that of the pressure by P. These four quantities are independent of the time. If the fluctuating or turbulent parts of the velocity components and of the pressure are denoted by u'_1, u'_2, u'_3, p', we shall have:

$$u_i = U_i + u'_i \; ; \; p = P + p' .$$

These expressions can be substituted into Eqs.(1) or (1a) and (2), which lead to the following results (for simplicity the primes have been omitted after the substitution, so that all small letters now indicate turbulent quantities):

$$\rho \left(\frac{\partial u_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} \right) =$$

$$= - \frac{\partial P}{\partial x_i} - \frac{\partial p}{\partial x_i} + \rho\nu \, \Delta \, U_i + \rho\nu \, \Delta \, u_i \; ;$$

$$\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0.$$

We take mean values of all terms occurring in these equations in order to obtain a set of equations governing the main flow. In this process, terms that are linear in the turbulent components drop out. The terms of the second degree in the turbulent components, however, will not drop out in general. We write:

$$(3) \qquad \tau_{ij} = -\rho\,\overline{u_i\,u_j}$$

The quantities so obtained are expressions for the momentum transfer due to the turbulent motion. They are analogous to similar expressions used in the kinetic theory of gases for the explanation of the major part of the viscous forces. In hydrodynamics they are known as the Reynolds' stresses.

The equations for the main flow can now be brought into the form:

$$(4) \qquad \rho\,U_j\frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho\nu\,\Delta\,U_i$$

$$(5) \qquad \frac{\partial U_i}{\partial x_i} = 0\ .$$

If these equations are subtracted from the full equations, we are left with a set of equations governing the turbulent motion. They are of more complicated type and can be written:

$$(6) \quad \rho\left(\frac{\partial u_i}{\partial t} + U_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} - \frac{\partial(\rho u_i\,u_j - \rho\,\overline{u_i\,u_j})}{\partial x_j} + \rho\nu\,\Delta\,u_i$$

$$(7) \qquad \frac{\partial u_i}{\partial x_i} = 0\ .$$

42. From Eq. (4) it will be seen that the Reynolds' stresses can be considered as a system of interior forces acting on the main flow in consequence of the presence of the turbulent motion. The component τ_{xz} of these stresses is the one we already encountered in Eqs. (18) and (18a) of the preceding chapter.

The equations are greatly simplified if we restrict ourselves to the case where the mean flow is in the direction of the x-axis only. It follows from the equation of continuity that the velocity component U_1 (for which we can simply write U) must be independent of x. We shall assume, moreover, that it is independent of y. We write $dU/dz = U'$; $d^2U/dz^2 = U''$. In this case we can assume that the turbulent motion is not only stationary with respect to time, but that, statistically speaking, it will also be independent of x and y. Hence, also the mean values of u, v, w, and p calculated with respect to x or y will be zero. The pressure P will depend linearly on x; $\partial P/\partial x$ will be a constant throughout the whole field; $\partial P/\partial y = 0$; $\partial P/\partial z$ will be independent of x and y.

Since there is no acceleration of the main motion, the left-hand side of Eq. (4) becomes zero. Derivatives of the Reynolds' stresses with respect to x and y drop out; and $\tau_{xy} = -\rho\,\overline{uv}$ and $\tau_{yz} = -\rho\,\overline{vw}$ both are zero from reasons of symmetry. Hence we are left with:

$$(8)\qquad \left\{\begin{aligned} 0 &= -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho\nu\, U'' \\ 0 &= -\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} \end{aligned}\right.$$

where $\tau_{xz} = -\rho\,\overline{uw}$ and $\tau_{zz} = -\rho\overline{w^2}$. The second equation is not very important. The first one shows that the Reynolds' stress τ_{xz} balances the combined effect of the average pressure gradient and of the viscous friction experienced by the main flow. Since in a large part of the field this viscous friction is exceedingly small, it will be evident that the Reynolds' stress τ_{xz} is the principal quantity determining the resistance experience by the main flow.

One of the major aims of turbulence theory is to find a method for calculating this stress directly from the equations governing the turbulent motion (Eqs. 6 and 7), taken together with appropriate boundary conditions, but without introducing assumptions on Lagrangian correlations or the like.

At present the theory is not yet developed far enough to make possible the execution of this program. Nevertheless it is of interest to give some time to an analysis of various features of the equations mentioned, since this will throw light on the character of the turbulent motion. Examples will be given in the next sections.

In the present case Eq. (6) takes the form:

$$\rho\left(\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+w\,U'\right) = -\frac{\partial p}{\partial x}-\frac{\partial}{\partial x_j}\left(\rho u_i\,u_j-\rho\overline{u_i\,u_j}\right)+\rho\nu\Delta u$$

$$(9)\qquad \rho\left(\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial y}-\frac{\partial}{\partial x_j}\left(\rho u_2\,u_j-\rho\overline{u_2\,u_j}\right)+\rho\nu\Delta v$$

$$\rho\left(\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right) = -\frac{\partial p}{\partial z}-\frac{\partial}{\partial x_j}\left(\rho u_3\,u_j-\rho\overline{u_3\,u_j}\right)+\rho\nu\Delta w$$

43. The Energy Equations for the Main Flow and for the Turbulent Field.

To obtain the energy equation for the main flow, in the simple case to which Eqs. (8) refer, we multiply the first equation of (8) by U; this gives:

$$0 = -U\frac{\partial P}{\partial x}+U\frac{\partial \tau_{xz}}{\partial z}+\rho\nu\;UU''.$$

The second and third terms on the right-hand side can be transformed into:

$$\frac{\partial}{\partial z}\left\{U(\tau_{xz}+\rho\nu U')\right\} - U'\tau_{xz}-\rho\nu(U')^2\ ,$$

and the equation can be rewritten:

$$(10)\qquad -U\frac{\partial P}{\partial x}+\frac{\partial}{\partial z}\left\{U(\tau_{xz}+\rho\nu U'\right\} = U'\tau_{xz}+\rho\nu(U')^2\,.$$

The first term on the left-hand side is the energy supplied to the field per unit volume and in unit time through the pressure gradient which maintains the main flow. The second term has the form of a derivative; this implies that it represents a transfer of energy from an element to adjacent ones. If we do not consider a single element of volume, but integrate the equation over a large domain, bounded by two planes z = constant at which U is zero, this term disappears. If U is not zero at the bounding planes, there can be transfer of energy to the flow from the outside, or inversely. But a term having the form of a derivative never represents a loss or a gain of energy in the interior of the field.

On the right-hand side we have first a term representing work done in connection with the Reynolds' stress τ_{xz}. This same term will turn up on the left-hand side of the energy equation for the turbulent motion. It indicates that there is a transfer of energy from the main motion to the turbulent motion. The second term on the right-hand side gives the energy dissipated as a consequence of the action of viscosity on the main motion.

Hence the equation states that energy derived from exterior sources is spent, partly in overcoming the Reynolds' stress and partly through viscous dissipation in the main flow. The first part is transferred to the turbulent motion.

In order to obtain the energy equation for the turbulent motion, we multiply the three equations of the system (9) by the corresponding components u_i and add. The result can be brought into the form:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\rho u_i^2}{2}\right) + \rho u w U' =$$

$$= -\frac{\partial}{\partial x_i}(p u_i) - \frac{\partial}{\partial x_j}\left(\frac{\rho u_i^2 u_j}{2}\right) + u_i \frac{\partial}{\partial x_j}\left(\overline{\rho u_i u_j}\right) + \rho \nu u_i \Delta u_i \; .$$

We take the mean value and obtain:

$$-U' \tau_{xz} = -\frac{\partial}{\partial x_i}\left(\overline{p u_i}\right) - \frac{\partial}{\partial x_j}\left(\overline{\frac{1}{2}\rho u_i^2 u_j}\right) + \rho \nu \overline{u_i \Delta u_i} \; .$$

The term $u_i \Delta u_i$ can be transformed in several ways. The one most commonly used is:

$$u_i \Delta u_i = \frac{\partial}{\partial x_j}\left(u_i \frac{\partial u_j}{\partial x_i} + u_i \frac{\partial u_i}{\partial x_j}\right) - \Phi \, ,$$

where

$$(12) \quad \Phi = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 .$$

Again several groups of terms in the energy equation have obtained the form of derivatives, meaning that these terms represent transfer of energy between adjacent elements of volume. They can be related to exchange of energy with outside sources through the surfaces bounding the field. In the case of motion through a tube with fixed walls and with turbulence independent of x, such an exchange does not take place. If we omit these terms, those remaining depend on the function Φ and represent the loss of energy through viscous dissipation in the turbulent motion. We can write the equation in the form:

$$(11a) \qquad U'\tau_{xz} = \rho\nu\,\Phi + \text{derivatives.}$$

It can then be interpreted as follows: on the left-hand side we have the energy derived from the main motion through the intermediary of the Reynolds' stresses; on the right-hand side we have the dissipation through viscosity. Finally there are transfer terms which do not play a part in the process when we consider the field as a whole.

44. The function Φ determining the loss of energy through viscosity is called the dissipation function. It has great importance in turbulence theory. It is possible to split off some further terms having the form of derivatives. In this way we can obtain:

$$(12a) \quad \bar{\Phi} = \gamma_x^2 + \gamma_y^2 + \gamma_z^2 + 4\left[\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right)\right] + \text{etc.},$$

where the γ's represent the components of the vorticity. In certain cases the latter form may be the most convenient; in others we use Eq. (12) for $\bar{\Phi}$.

The fact that the terms deriving from the turbulent pressure occur in the form of derivatives in Eq. (11) proves that they only bring about a transfer of energy, but do not lead to actual losses.

The same implies to the group of terms containing the turbulent velocity components to the third power which were derived from a multiplication of the nonlinear terms in the Navier-Stokes equations with the components of the turbulent velocity. It follows that whenever we attempt to construct a simplified system of formulas, which in their energy relations should be equivalent to the complete system, care must be taken that expressions substituted for turbulent pressures or for nonlinear terms satisfy similar conditions.

Although the nonlinear terms in the Navier-Stokes equations do not directly represent a loss or gain of energy, they nevertheless play an important part in promoting dissipation in an indirect way. The effect of these terms on the motion is to steepen velocity gradients in certain parts of the field, which brings about a local intensification of the dissipation.

Velocity gradients calculated from the mean amplitude of the turbulent motion and the average dimensions of the most conspicuous forms of eddy motion, are far too small to lead to the dissipation of energy needed to balance the inflow of energy from outward sources. The really important dissipation takes place in narrow regions or layers in which high velocity gradients have been produced by some process of concentration. It is also this process which ultimately determines the magnitude of the turbulent velocity components.

45. In order to obtain a picture of the way in which narrow regions of dissipation may be produced, we consider a simplified system of equations of motion. We start from the equations for v and w (u_2, u_3);

$$\rho \left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho\nu\Delta u$$

$$\rho \left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial (p - \tau_{zz})}{\partial z} + \rho\nu\Delta w\,.$$

In these equations we shall neglect the circumstance that U is a function of z and treat U as a constant; we can then introduce a moving coordinate system and take up the derivatives $U(\partial v/\partial x$, $U(\partial w/\partial x)$ into the time derivative. We further neglect the terms $u(\partial v/\partial x)$, $u(\partial w/\partial x)$; $\nu(\partial^2 v/\partial x^2)$, $\nu(\partial^2 w/\partial x^2)$, partly in connection with the error already introduced by taking U constant, partly on the assumption that on the whole derivatives with respect to x will be of a smaller order of magnitude than derivatives with respect to y or z. Since now u no longer occurs in the equations, we need not introduce the equation of continuity, and we omit all restrictions on the value of $\partial v/\partial y + \partial w/\partial z$. In connection with this we shall neglect the pressure terms. We are then left with:

$$(13) \qquad \begin{cases} \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = \nu\left(\dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}\right) \\ \\ \dfrac{\partial w}{\partial t} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = \nu\left(\dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}\right) \end{cases}$$

So long as no steep gradients of v and w have appeared, we moreover can neglect the viscosity and retain:

$$(13a) \qquad \begin{cases} \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = 0 \\ \\ \dfrac{\partial w}{\partial t} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = 0 \end{cases}$$

Although the systems (13) and (13a) represent a very mutilated form of the Navier-Stokes equations, they can serve to demonstrate the tendency for steepening of velocity gradients. The system (13a) possesses characteristics determined by the relations:

$$(14) \qquad dy/dt = v; \quad dz/dt = w.$$

Along a characteristic we have:

$$(15) \qquad dv/dt = 0; \quad dw/dt = 0,$$

which means that the velocity components are propagated along the characteristics without change. This is, of course, a consequence of neglecting the pressure effects. It follows that the characteristics must be straight lines.

In principle, this result makes it possible to derive any subsequent state of the field from an initially given state. Difficulties, however, arise when characteristics meet each other. Without going into the theory, we can expect that there will be a marked difference between regions where the motion is divergent and regions where it is <u>convergent</u>. In regions where there is convergence, velocities of different magnitudes and direction will be brought close together and we must expect that steep gradients will be produced.

We will not go into an analysis of this phenomenon but mention that it can be proved in a relatively simple way if one assumes that the initial state of the field is formed by a patchwork of domains, in each of which v and w are linear functions of y and z. The equations can then be completely integrated and it is found that regions of convergence usually contract into a segment of a straight line. This line becomes the seat of a discontinuity of the velocity, in general both for the normal and for the tangential component.

This result, however, requires correction in two ways. In the first place, as soon as steep gradients appear the viscosity terms of the original equations (13) must be taken into account. These terms prevent the appearance of mathematical discontinuities and turn them into transition layers, with large, but finite gradients. When no other effects are taken into account, it is found that the dissipation in these transition layers obtains a finite value, independent of the magnitude of the viscosity and given by certain expressions of the third degree in the velocity differences across the layer. This is the main feature in the influence of nonlinear terms on the dissipation of energy.

The other correction is necessary because we have neglected the equation of continuity. Convergence of flow in the y, z - plane towards a narrow region requires a large positive value of $\partial u/\partial x$. Such a value cannot appear without affecting the pressure distribution, and the latter will react on the breadth of the region of convergence. Since in regions of divergence steep gradients do not appear, only relatively small negative values of $\partial u/\partial x$ are required by these regions and no marked pressure effect is to be expected in them.

There is a further question to be investigated: what happens to a layer of convergence after it has been formed? In general, such a layer will not remain at a fixed position of the field; it will be displaced and usually it will obtain a curved form. It is possible that different layers meet each other; there is a probability that they will flow together in such a case.

Some of these points can be investigated by making a still farther reaching simplification in which we restrict to one coordinate and one component of velocity, to which we will come later on.

46. Even without further refinement, the result we have arrived at is of interest in connection with experimental evidence.

The most important point is the difference existing between uni-axial extension and uni-axial compression of regions of the field. These two phenomena, although having opposite effects in their incipient stages, present a markedly different character in their further development. The results of an extension on the whole are not reversible.

One consequence is the tendency of a turbulent field to divide itself into a number of separate regions separated from each other by thin transition layers formed through convergence of the flow in the y, z-plane. This is of importance in connection with the observed result, that in turbulence elements of volume with different values of the velocity seem to follow each other in an irregular way, with very thin transition layers separating them, as had been mentioned in Section 20.

When we do not restrict ourselves to the consideration of the flow in a y, z-plane, but give attention to the x- coordinate, the line segments towards which the convergence occurs will appear as the section of flat ribbons extending in the direction of the main flow. We can expect that these ribbons will curl about the streamlines of the main flow. They may start at some place and may coalesce with other ribbons, or may, presumably, also disappear at some place further downstream.

Such ribbon-like features are often observed when foreign matter is brought into the flow. An interesting example is to be seen when a strong wind blows over a sandy plane or beach. The sand taken up by the wind moves in thin layers or ribbons, constantly shuffling to and fro, and folding and curling about the streamlines.* Similar ribbons can sometimes be seen in experiments with water channels. Flames, the curling veils or ribbons of smoke which rise from a lighted cigarette, and veils of vapor rising from a hot liquid belong to the same class of phenomena, since in all of them a

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*R. A. Bagnold, "The Physics of Blown Sand and Desert Dunes", London, 1941, p. 176-179.

certain convergence of the flow plays a part.

There exists a problem whether, in the type of turbulent motion we are considering, vorticity with the axis of rotation more or less parallel to the direction of the main flow can be preponderant over vorticity with axes of rotation directed transversely. (Since we have assumed that U was a function of z, there will be, of course, in the field a steady vorticity depending on dU/dz.)

The question of the preponderance of longitudinal over transverse vorticity has been raised in connection with certain theories of turbulent motion. It has been pointed out, e.g., by S. Goldstein*, that vortex motion with the axis of rotation parallel or perpendicular to the direction of the main flow is responsible for the difference between the so-called "momentum transfer" theory proposed by Prandtl to explain the mechanism underlying turbulent friction; and "vorticity transfer", which was the mechanism considered by G. I. Taylor. Vorticity transfer seems to occur in cases where vortex motion with axes perpendicular to the main flow is produced very intensively, as is the case in the flow along a long cylindrical obstacle transverse to the main stream. Momentum transfer by "longitudinal" vortices appears to be a feature governing boundary layer flow.

We may add that longitudinal vortices appear always along the walls of tubes or canals when an obstacle is present or when there is a bend in the tube or canal. In these cases the flow pattern has a relatively stable form and leads to what is called "secondary flow"; this does not properly constitute turbulence, but the two types of flow seem to be closely connected. Apparently there is always a tendency to form longitudinal vortices in the neighborhood of walls. When the wall is smooth, these vortices do not have a stable position but will constantly waver about, thus constituting part of the turbulence. When there are certain obstacles, or the like, which stabilize these vortices, they appear as secondary flow. This point of view may bring into connection, for instance, the various opinions brought forward in the discussion of V. A. Vanoni's paper (already quoted).**

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* "Modern Developments in Fluid Dynamics", Oxford 1938, Vol. I, pp. 206-213.

** "Transportation of Sediment by Water", Trans. Amer. Soc. C.E. 111, pp. 67-133, 1946.

47. The Increase of Dissipation Caused by the Concentration of Vorticity.

In the simplified example mentioned in Section 45, it was found that in regions where $(\partial v/\partial y + \partial w/\partial z) < 0$, there is a tendency to concentrate vorticity into narrow sheets or ribbons. In that example the effect is obtained in a very marked way, owing to the circumstance that the pressure has been neglected. Attempts to take account of the pressure effects in a simple way so far have not been successful. What evidence could be obtained, rather pointed to a much smaller concentration of vorticity and to vortex sheets having a thickness proportional to $\sqrt{\nu}$. A greater concentration, however, can be obtained if we consider concentration towards a line, such as is found when a vortex tube is extended axially. It was G. I. Taylor who first pointed out that the longitudinal extension of vortex tubes must be the main factor in turbulent dissipation, and who also observed that extension must occur more frequently than shortening, since turbulent motion has a diffusive character, so that elements of volume which originally were neighbors will usually tend to move apart.*

It is not difficult to calculate what is the ultimate result that can be obtained when a rectilinear symmetrical vortex is extended longitudinally at a constant rate.** We consider an axially symmetric field with velocity components u (parallel to the x-axis, which is the axis of the field); v (tangential) and w (radial). It is assumed that:

$$(16) \qquad \ldots \quad u = 2Ax; \quad w = -Ar,$$

while v shall be a function of r and t which must be found.

In this case there is only one component of vorticity:

$$\gamma_x = \frac{1}{r}\frac{\partial(rv)}{\partial r}$$

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*G.I. Taylor, Journ. Aeron. Sciences 4, p. 315, 1937; G.I. Taylor and A. E. Green, Proc. Roy. Society London, A, 158, p. 501, 1937; G.I. Taylor, Proc. Roy. Society London, A. 164, p. 15, 1938; also S. Goldstein, "Three-Dimensional Vortex Motion in a Viscous Fluid," Philos. Mag., (VII) 30, p. 85, 1940.

**J. M. Burgers, Proc. Acad. Sciences Amsterdam 43, p. 11, 1940.

We assume that the pressure is given by:

$$p = -\frac{\rho A^2}{2}(4x^2 + r^2) + \rho \int dr\, \frac{v^2}{r} \; .$$

Such a pressure field cannot extend over large distances; the expression should be considered as an approximation over a small region containing a certain length of the x-axis.

The equations of motion in cylindrical coordinates have been given by S. Goldstein.* It will be found that the expressions for u, w, p, given above, satisfy the equation of continuity and the equations of motion for the axial and radial directions. There remains the equation of motion for v, which has the form:

$$\frac{\partial v}{\partial t} - Ar\frac{\partial v}{\partial r} - Av = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \; .$$

We will look for a solution independent of the time, assuming that this may represent a state which is asymptotically approached. The resulting equation can be written:

$$Ar\frac{d(rv)}{dr} + \nu\left\{\frac{d^2(rv)}{dr^2} - \frac{1}{r}\frac{d(rv)}{dr}\right\} = 0 \, ,$$

and has the solution:

(17) .... $$v = \frac{C}{2\pi r}\left(1 - e^{-Ar^2/2\nu}\right) ,$$

with:

(17a) .... $$\gamma_x = \frac{AC}{2\pi\nu} e^{-Ar^2/2\nu} ,$$

C being the integration constant. It will be seen that vorticity is to be found only in a narrow cylindrical space surrounding the x-axis; the strength of the vortex (circulation along a curve encircling it) is equal to C.

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* "Modern Developments in Fluid Dynamics" (Oxford, 1938), Vol. I, pp. 103-104.

To find the dissipation, we must calculate the dissipation function as given by (12). This requires transformation from the cylindrical coordinates to rectangular ones. The result appears to be of the form:

$$\Phi = 12A^2 + \frac{C^2}{\pi^2 r^4}\left(1 - e^{-Ar^2/\nu}\right)^2 + \frac{C^2A^2}{4\pi^2\nu^2}\, e^{-Ar^2/\nu} + \text{terms contain-}$$

ing exponential factors and having factors ν^{-1} or ν^0. We must multiply Φ by $\rho\nu$ and integrate over the domain to which we apply our solution. The result contains certain terms depending on the magnitude of this volume, which, however, are multiplied by ν and a further term independent of ν :

$$(18) \quad \dots\dots \quad \rho\, \frac{C^2A}{4\pi} \quad . \quad \text{(length of the vortex)} .$$

Hence the resulting dissipation is practically independent of the viscosity. It depends on the number and the strengths of the vortex tubes, and on the absolute value of the difference of the longitudinal velocities u_{II}, u_I at the ends of the extended part. This is seen if we remember that $u = 2Ax$, so that (18) can be written:

$$(18a) \qquad \begin{matrix}\text{dissipation in an}\\ \text{extended vortex}\\ \text{tube}\end{matrix} = \rho\, \frac{(\text{circulation})^2}{8\pi} \cdot \left| u_{II} - u_I \right| .$$

48. Note on the problem of "Momentum Transfer" versus "Vorticity Transfer".

We return to the first of Eqs. (8) for the main flow. The term $\partial\tau_{xz}/\partial z$ in this equation represents what is left from the more complete expression for the Reynolds' stresses which should be written:

$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}$$

or:

$$-\frac{\partial}{\partial x}(\rho\overline{u^2}) - \frac{\partial}{\partial y}(\rho\,\overline{uv}) - \frac{\partial}{\partial z}(\rho\,\overline{uw}) .$$

Now this group of terms can be transformed in a different way, introducing the components of vorticity, giving

$$- \frac{\partial}{\partial x} \left[\frac{\rho}{2} \overline{(u^2 + v^2 + w^2)} \right] + \rho \, \overline{(v\gamma_z - w\gamma_y)} \; .$$

In the case considered here, where the turbulence is the same for all values of x, the first term drops out and the influence of the turbulent motion on the pressure gradient of the main motion is given by:

$$(19) \quad \ldots . \qquad \rho \, \overline{(v\gamma_z - w\gamma_y)}$$

In order to obtain an estimate of this quantity, the concept of the "mean free path" has been used (see Section 40). For a more complete treatment we refer to S. Goldstein.* It is assumed that an element of fluid passing a point P (coordinates x, y, z) at the instant t, originally has made itself free from surroundings at the point a, b, c, where:

$$a = x - \ell_1 \; ; \quad b = y - \ell_2 \; ; \quad c = z - \ell_3 \; .$$

When the velocity U of the main motion is a function of z only, this element, when at a, b, c, had the mean vorticity existing there with components:

$$\overline{\gamma}_x = 0 \; ; \quad \overline{\gamma}_y = U' - \ell_3 \, U'' \; ; \quad \overline{\gamma}_z = 0 \; .$$

When the element has arrived at P (x, y, z), it will have taken its vorticity with it, but the vorticity vector may have turned and have been extended, dependent on the motion of the element. The components of vorticity with which it arrives at P will be given by:

$$\gamma_x = \overline{\gamma}_y \frac{\partial x}{\partial b} \; ; \quad \gamma_y = \overline{\gamma}_y \frac{\partial y}{\partial b} \; ; \quad \gamma_z = \overline{\gamma}_y \frac{\partial z}{\partial b} \; .$$

The expression (19) consequently takes the form:

$$(20) \quad \ldots . \; \rho \; \overline{(U' - \ell_3 \, U'') \left(v \frac{\partial z}{\partial b} - w \frac{\partial y}{\partial b} \right)}$$

Two particular cases can now be considered separately. First assume that all turbulent motions are confined to the x, z-plane and are independent of y,

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* "Modern Developments in Fluid Dynamics", (Oxford, 1938), Vol. I, pp. 205-214, and literature quoted there.

as would be the case with purely two-dimensional turbulence. The only vorticity component possible then is γ_y. We have:

$$\frac{\partial u}{\partial y} = 0, \quad v = 0, \quad \frac{\partial w}{\partial y} = 0,$$

from which it follows that:

$$\frac{\partial x}{\partial b} = 0; \quad \Big| \quad \frac{\partial y}{\partial b} = 1; \quad \Big| \quad \frac{\partial z}{\partial b} = 0.$$

The expression (20) consequently changes into:

$$- \rho U' \overline{w} + \rho U'' \overline{\ell_3 w} .$$

The first term drops out, since $\overline{w} = 0$. If we write ℓ for ℓ_3, there remains:

$$(21) \quad \dots \quad + \rho \frac{d^2U}{dz^2} \overline{\ell w} .$$

This expression had been obtained by G. I. Taylor as a result of the "vorticity transfer theory" (1915).

Next, suppose that the turbulent motion is of such a nature that $\partial w/\partial x$, $\partial v/\partial x$, and $\partial w/\partial x$ can be put equal to zero. We then have:

$$\partial v/\partial y + \partial w/\partial z = 0,$$

from which it follows that:

$$\partial \ell_2/\partial y + \partial \ell_3/\partial z = 0.$$

We then have:

$$\frac{\partial y}{\partial b} = 1 + \frac{\partial \ell_2}{\partial y} = 1 - \frac{\partial \ell_3}{\partial z}; \quad \frac{\partial z}{\partial b} = \frac{\partial \ell_3}{\partial y} .$$

Expression (19) now takes the form:

$$\rho\,\overline{(U' - \ell_3 U'')\left(v\frac{\partial \ell_3}{\partial y} - w + w\frac{\partial \ell_3}{\partial z}\right)} =$$

$$= \rho\,U''\,\overline{\ell_3 w} + \rho\,U'\,\overline{\left(v\frac{\partial \ell_3}{\partial y} + w\frac{\partial \ell_3}{\partial z}\right)} -$$

$$- \frac{1}{2}\,\rho\,U''\,\overline{\left(v\frac{\partial v\ell_3{}^2}{\partial y} + w\frac{\partial w\ell_3{}^2}{\partial z}\right)} \;.$$

Here $$\overline{v\frac{\partial \ell_3}{\partial y}} = -\,\overline{\ell_3\frac{\partial v}{\partial y}} = +\,\overline{\ell_3\frac{\partial w}{\partial z}} \;.$$

Further $$\frac{\partial}{\partial y}\,\overline{v\ell_3{}^2} = Q\;; \qquad \frac{\partial}{\partial z}\,\overline{w\ell_3{}^2}$$ is neglected.

In this way there remains:

$$(22)\; \ldots\ldots \quad \rho\,U''\,\overline{\ell_3 w} + \rho U'\,\frac{\partial}{\partial z}\,\overline{\ell_3 w} = \rho\,\frac{d}{dz}\left(\frac{dU}{ds}\,\overline{\ell w}\right).$$

The two assumptions used, that of turbulence being confined to motions in the x, z-plane, and that of motions confined to the y, z-plane, are in some way connected with different assumptions about the effect of turbulent pressure fluctuations. Formula (22) is obtained from the theory of momentum transfer, in which it is assumed that an element of volume takes its u-velocity with it over a free path, uninfluenced by pressure fluctuations. In the vorticity transfer theory this assumption is not introduced, but it is supposed that the strength of a vortex with the axis parallel to the y-axis is unaffected by the transport over the free path.

49. Experimental evidence has shown that in cases where turbulent motion is superposed on a main motion, as in the flow through a tube, the velocities of the turbulence usually are only a few percent of the velocity of the main motion. This has often suggested that perhaps certain characteristic features of turbulence could be deduced from simplified equations of motion in which terms of the second degree in u, v, w have been omitted.

Such equations could show the effect of the coupling between the main motion and the turbulent motion. The coupling between the various components of the turbulent motion itself will not be obtained by such a method. Hence the results certainly will not be applicable at the far end (the small wavelength end) of the spectrum. It is doubtful also whether the results will have much meaning at the long wavelength end, for it appears that the aggregate effect of all the small wavelength components is of great importance even here. It is this circumstance which makes it difficult to arrive at definite results about the long wavelength components from linearized equations. It has been proposed in certain investigations to represent the aggregate effect of the small wavelength components as an increased viscosity, but the magnitude of the coefficient to be used is not known.

The linearized equations for the turbulence are identical with the equations used in investigations on the stability of laminar flow. They are obtained from Eq. (9) of Section 42, and have the form:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + wU' = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u$$

$$(23)\qquad \begin{aligned} \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\Delta v \\ \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta w \end{aligned}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 .$$

It is assumed that U is a given function of z. Solutions of this system of equations have been derived by putting:

$$(24)\qquad \begin{aligned} u &= \phi_1(z)\cdot e^{i(\alpha x + \beta y + \omega t)} \\ v &= \phi_2(z)\cdot e^{\cdots} \\ w &= \phi\ (z)\cdot e^{\cdots} \\ p/\rho &= P\ (z)\cdot e^{\cdots} \end{aligned}$$

where ϕ_1, ϕ_2, ϕ and P represent unknown functions of z. When these expressions are substituted into (23), the functions ϕ_1, ϕ_2 and P can be eliminated. The remaining equation for ϕ is:

$$(U - \frac{\omega}{\alpha}) \left\{ \phi'' - (\alpha^2 + \beta^2)\phi \right\} - U''\phi + \frac{i\nu}{\alpha} \left\{ \phi^{IV} - 2(\alpha^2+\beta^2)\phi'' + (\alpha^2+\beta^2)^2 \phi \right\} = 0 . \tag{25}$$

This is the standard form for the investigation of the stability of laminar motion. Numerous papers have been devoted to this subject; for a summary refer to C. C. Lin.* Usually β is taken as zero, which does not change the essential character of the equation.

When viscosity is small, the terms multiplied by ν can be neglected in those parts of the field where there are no sharp gradients in ϕ. It is possible, however, that the factor $U - \omega/\alpha$, which multiplies the first term of the equation, vanishes for one or more values of z. If ω is real, the ratio ω/α gives the velocity of propagation of the disturbances in the direction of the x-axis, and a singularity appears where this velocity is equal to the velocity of the main flow. This circumstance has played a big part in all investigations, where the crucial point was to obtain solutions with real ω, separating the domain of damped solutions from that of increasing (unstable) solutions. Reintroduction of the terms with ν (at least of the most important term $i\,\phi^{IV}/\alpha$) prevents the appearance of a logarithmic infinity in ϕ, but a concentrated layer of vorticity is obtained in the neighborhood of one or more critical values of z.

Since the mathematical theory involved in these relations is very complicated, it does not look promising at the present moment to continue in this direction and to investigate the still more difficult question of the coupling between the various possible components. In the next chapter we shall turn to a simplified mathematical model, which will help us to obtain insight into certain aspects of the coupling problem.

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* "On the Stability of Two-Dimensional Parallel Flow," Quart. Appl. Math. 3 (1945-46), pp. 117-142, 218-234, 277-301.

CHAPTER VII

Application of a Mathematical Model to Illustrate Relations Characteristic of Turbulence

There is no method to obtain exact solutions of the Navier-Stokes equations which will give the complete history of a field of flow for all $t > 0$ when the initial state of the field at $t = 0$ has been specified. Even investigations concerning the existence of such solutions have not come to a final result. We must add that from the physical point of view, complete knowledge of a way for constructing an exact solution starting from fully specified initial data would not even have much interest. The calculations required would be extremely laborious and complicated. Each case treated would depend in an extremely sensitive way on every detail of the initial conditions and, moreover, the labor spent would bring information of an incidental character only, in which features of general interest would be hard to recognize.

What is needed would be something different, viz., a method for obtaining averages and statistical properties referring to a whole class of solutions, the class being defined by a general type of initial conditions, specified likewise in a statistical way only. For instance, the initial data might refer to mean values of the velocity and the pressure, together with mean amplitudes of the fluctuations. Instead of initial conditions, boundary conditions can be given for problems of stationary flow. Thus, in the case of flow through a pipe, the mean velocity of flow over the cross section could be given, together with the mean amplitudes of velocity and pressure fluctuations at the entrance of the pipe and statistical data concerning the roughness of the walls. It will be the ultimate aim of turbulence theory to find methods for obtaining the statistical properties of classes of solutions.

As yet there is no indication of the way in which such methods should be found. All modern theories of hydrodynamic turbulence make only a partial use of the Navier-Stokes equations and attempt to supplement them by introducing auxiliary hypotheses. Sometimes the latter are deduced from dimensional considerations. In other cases guesses are made concerning the probable form of certain functions, or certain probability hypotheses are introduced in which, for instance, the velocity is treated as a random variable. In all these theories there remain arbitrary elements. Although a number of results deduced from them seem to be well confirmed by experimental research, the introduction of auxiliary assumptions, which could not be based directly on the Navier-Stokes equations themselves, represents an unsatisfactory element.

The problem of turbulence is an instance of a much wider class of mathematical problems which refer to statistical properties of classes of solutions of nonlinear equations. It has, therefore, been considered to be of interest to show that various of the mathematical questions which turn up in relation to turbulence, likewise apply to a much simpler nonlinear equation. This equation presents a character comparable to that of the Navier-Stokes equations. Although it refers to a single variable only (instead of three velocity components and the pressure), it has retained the features defining energy transfer and dissipation. Solutions of this equation starting from specified initial conditions can be obtained, either by means of exact methods, or with the aid of a very convenient method of approximation. Classes of solutions can be defined, depending on initial conditions or on boundary conditions which are specified in a statistical way only, and the study of these classes leads to statistical problems in many ways analogous to statistical problems occurring in turbulence. Also, many of the methods of approach used in hydrodynamic turbulence theory can be applied to the solutions of this equation. The equation can therefore procure material which can aid in testing various auxiliary hypotheses.

It has, therefore, been considered worthwhile to give attention to this equation and its solutions, and to use it as an introduction into some of the deeper turbulence problems.

50. The equation to be studied has the form:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} \tag{1}$$

The variable v can be taken as the analogue of the turbulent velocity. It is dependent on the time and a coordinate y (something like the transverse dimensions of the field of flow in a tube). The equation has a nonlinear term of the first order, and a term of the second order multiplied by a coefficient ν which is assumed to be very small (analogue of the kinematic viscosity). The interplay of these terms is of primary importance in the dynamics of turbulence. Since the equation refers to a single variable and a single coordinate, it does not picture those geometric relations of hydrodynamic turbulence which are dependent on the three-dimensional nature of the field, the properties of vortices and the phenomena of shearing motion. There is no pressure term in the equation and there is no equation of continuity; hence there is nothing which reflects the condition of incompressibility of ordinary hydrodynamics. In a way, the equation is more illustrative of certain phenomena peculiar to compressible media, in particular to shock waves. In the results which can be obtained from Eq. (1) there is no approach to the special geometric features (tensor relations, etc.) which are considered in the theory of isotropic turbulence.

The equation can be completed on the right-hand side by introducing either a term representing an exterior force, or a term describing a coupling between v and another parameter, to be considered as the analogue of the velocity of the main motion. It is necessary to specify the domain of the variable y to which the equation shall refer. This can be either the complete y-axis, from $-\infty$ to $+\infty$, or it can be a limited domain, for instance $0 \leq y \leq b$, in which case we shall require that v vanishes at both limits.

Equation (1) can be reduced to a linear equation by substituting a new variable for v. It is useful, however, first to consider the equation as it stands. We look for a particular solution of the form:

$$(2) \qquad v = \beta\,(y - \sigma)\,,$$

where β and σ are functions of t. When this expression is substituted into (1), the equation is satisfied identically if

$$(3) \qquad \beta = 1/(t - t_0)\,; \qquad \sigma = \text{constant}$$

t_0 being a constant. Hence a solution represented by a straight line will turn to the right, while its point of intersection with the axis (σ) remains unchanged. The angle α between the segment and the vertical direction increases according to the relation:

$$\tan \alpha = t - t_0\,.$$

The result can be extended to the case where the course of v is represented by a series of straight segments. To prevent difficulties with quantities becoming infinite, we assume that at the intersection point between consecutive segments there is a slight rounding off of the peak. Since $\partial^2 v/\partial y^2$ is not zero in the neighborhood of the angles, the term $\nu(\partial^2 v/\partial y^2)$ will have influence on the form of the solution. Keeping in mind the effect of a similar term in the equation of diffusion, we may expect that the rounding off will gradually spread to a more extended region. In the first phases of the motion this detail will not have much influence and we can find the approximate history of the solution simply by applying the rule that every segment turns about its "hinge point", δ, according to formulas (2) and (3). The point of intersection of two consecutive segments retains a constant value of v and moves with a velocity equal to v. If the point is below the axis, v is negative and the movement is directed to the left.

51. When the initial slope of the segment is positive, it will remain so with the slope gradually decreasing to zero. If the initial slope is negative, the slope will increase in absolute measure and after a finite lapse of time the segment will approach to a vertical position. Our simple result can then no longer be used.

To see what will happen in such a case, we consider a particular solution of (1), obtained by assuming that v is determined by an expression of the form:

$$v = V(\eta)\ (t - t_o)^{-1/2}$$

where $V(\eta)$ is a function of the variable $\eta = (y - \sigma)(t - t_o)^{-1/2}$.

Equation (1) is then transformed into:

$$\nu V'' - VV' + \frac{1}{2}\eta V' + \frac{1}{2} V = 0$$

which can be integrated and gives:

$$\nu V' - \frac{1}{2}V^2 + \frac{1}{2}\eta V = 0\,,$$

(the integration constant has been adjusted so that V can be made to vanish at infinity). We now put:

$$V = -2\nu\, d(\ln u)/d\eta\,,$$

and obtain a linear equation for u:

$$u'' = -(\eta/2\nu) \cdot u' .$$

Integration of the latter equation finally leads to the following expression for V:

$$V = \frac{2\nu\, h \exp\left\{(h^2 - \eta^2)/4\right\}}{4\nu - h \displaystyle\int_h^{\eta} d\eta_1 \exp\left\{(h^2 - \eta_1^2)/4\nu\right\}}$$

h being another integration constant. Since this expression looks rather involved, it is useful to note the following approximations, valid when h^2/ν is large:

(a) when $\eta = h + \delta$ where $\delta \ll h$,

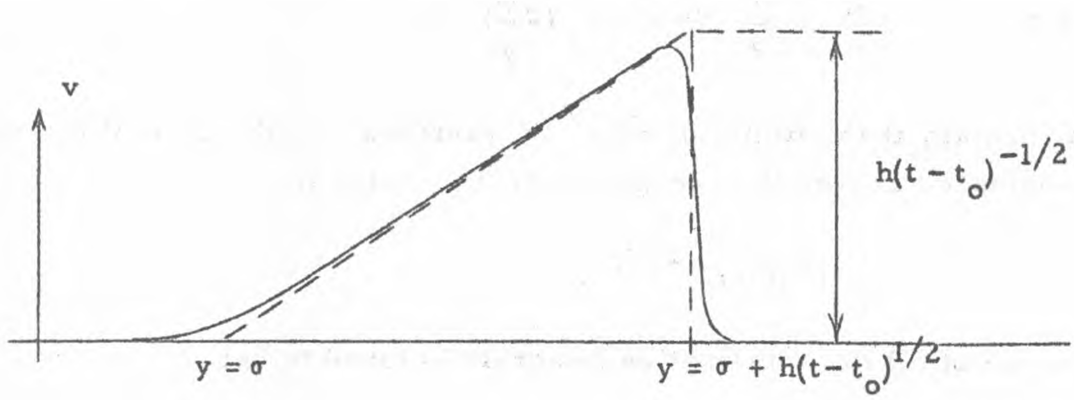
$$V \cong \frac{1}{2} h \left[1 - \tanh\ (h\delta/4\nu) \right] \quad ;$$

(b) when $0 < \eta < h$ and η not too near one of the end points:

$$V \cong \eta ;$$

(c) when η is near zero or is negative the value of V becomes practically zero.

With the aid of the approximations we can obtain a general picture of the course of V. This can be readily translated into a picture for v as a function of y and t, leading to the result given in the diagram.



It will be seen that the course of v is represented approximately by a triangular figure with constant area $\frac{1}{2}h^2$. The slope of the hypotenuse is $1/(t-t_o)$ as before. The velocity of advance of the right-hand side (the "front velocity") is $\frac{1}{2}h(t-t_o)^{-1/2}$, which is equal to one-half the height of the front.

An approximation to the course of v at the front is given by:

$$(4) \qquad v = \frac{h}{2\sqrt{t-t_o}}\left[1 - \tanh \frac{h(y-\xi)}{4\nu\sqrt{t-t_o}}\right]$$

with $\xi = \sigma + h(t-t_o)^{1/2}$.

It should be noted that there is also a solution in which the signs of both V and η are changed. In this case the triangle points downward and its front moves to the left.

52. The result that the area of the triangle is constant expresses the "conservation of momentum" for the solution. By integrating Eq. (1) with respect to y we obtain

$$(5) \qquad \frac{d}{dt}\int v \, dy = 0 ,$$

for any domain at the limits of which v vanishes and $\nu(\partial v/\partial y)$ is either rigorously zero or sufficiently small to be negligible.

If we multiply Eq. (1) by v and integrate, we obtain an "equation of energy"

$$(6) \qquad \frac{d}{dt}\int \frac{v^2}{2}\, dy = -\nu \int \left(\frac{\partial v}{\partial y}\right)^2 dy ,$$

for any domain at the limits of which v vanishes. In the case of the solution considered above, the energy integral amounts to:

$$\frac{1}{6}\, h^3 (t-t_o)^{-1/2} .$$

Making use of (4) the "dissipation integral" is found to be:

$$\frac{1}{12}\, h^3 (t-t_o)^{-3/2} .$$

It is easily verified that these expressions satisfy (6).

The result that a nearly vertical front in the course of v can be described approximately by a hyperbolic tangent function and that it has a certain velocity of advance can be generalized. We return to Eq. (1) and introduce a coordinate system which shall move with the front. To this end we put:

$$y' = y - \xi\,; \quad t' = t\,,$$

ξ being a function of t. From these formulas we deduce:

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}\,; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - c\,\frac{\partial}{\partial y'}\,.$$

where $c = d\xi/dt$. Substitution of these expressions into (1) gives:

$$\frac{\partial v}{\partial t'} + (v - c)\,\frac{\partial v}{\partial y'} = \nu\,\frac{\partial^2 v}{\partial y'^2}\,.$$

We expect that the derivative $\partial v/\partial y'$ will be of the order ν^{-1} in the region with a steep gradient, and that $\partial^2 v/\partial y'^2$ will be of the order ν^{-2}. If c is properly adjusted, $\partial v/\partial t'$ will be of normal order of magnitude, both at the steep front and elsewhere. This term can then be discarded and there remains:

$$(v - c)\,\frac{\partial v}{\partial y'} = \nu\,\frac{\partial^2 v}{\partial y'^2}\,.$$

Integration gives:

$$\frac{1}{2}(v - c)^2 - \nu\,\frac{\partial v}{\partial y'} = \text{constant.}$$

Since this expression must be valid through the region of a rapid change of v and also on both sides of it where $\partial v/\partial y'$ returns to a normal order of magnitude and $\nu(\partial v/\partial y')$ can be neglected, we find:

$$\frac{1}{2}(v_{I} - c)^2 = \frac{1}{2}(v_{II} - c)^2 = \text{constant}\,,$$

where the subscripts I and II denote the values of v on the two sides of the zone of rapid change. From this:

$$c = \frac{1}{2}(v_{I} + v_{II}) \tag{7}$$

Hence the velocity of advance of a steep front is given by half the sum of the values of v at both ends. A further integration gives:

$$(8) \qquad v = \frac{1}{2}(v_I + v_{II}) - \frac{1}{2}(v_I - v_{II}) \tanh \frac{(v_I - v_{II})(y - \xi)}{4\nu} .$$

By making use of this result we can now construct the development of v from any initial state given by a chain of straight segments, at least so long as the rounding off of the angles in consequence of the viscosity has not proceeded too far. Every segment turns about its "hinge point" according to the Eqs. (2) and (3), and when a downward sloping segment reaches the vertical position it does not turn any further but remains vertical, obtaining a velocity of advance given by Eq. (7). This velocity of advance will not be constant, since the values of v_I and v_{II} usually will change in course of time.

It is possible that consecutive vertical segments overtake each other. When this occurs we combine them to form a single segment, moving from then on with a velocity again given by (7), provided v_I and v_{II} refer to the velocities at the ends of the new segment.

It is found that every vertical segment is the seat of dissipation of energy in the amount $(v_I - v_{II})^{3/12}$. (It should be observed that $v_I - v_{II}$ is positive for every vertical segment, as will be evident from the process by which these segments are generated).

All these features can be checked by making use of the exact solution of Eq. (1), which can be obtained, as was indicated by J. D. Cole and V. Bargmann, by making the substitution:

$$v = -2\nu \, \partial(\ln u)/\partial y,$$

which is similar to the one used to transform the ordinary (not partial) differential equation for V.

53. One may ask what these considerations have to do with the turbulence problem.

The first point, already noted, is that Eq. (1) has the two important features of the hydrodynamic equations: a typical nonlinear term of the first order and a linear term of the second order multiplied by a small coefficient. It is possible to apply the similarity theory to Eq. (1) and it is found that its solutions are characterized by a Reynolds number formed by the product of a typical velocity and a typical length divided by the kinematic viscosity ν . Moreover, the solutions of Eq. (1) which vanish at infinity satisfy the condition of conservation of momentum, and an energy equation can be formed expressing the loss of energy through dissipation. (The nonlinear term of Eq. (1) vanishes both from the momentum and from the energy equation).

The presence of the nonlinear term has the consequence that features of the solutions are propagated with a velocity given by the magnitude of v itself, which has the further consequence that steep fronts can be generated which, once formed, keep their individuality so long as they do not merge with a neighboring front. This property can be compared with the tendency found in fluid motion that masses, having acquired a certain velocity, displace themselves with this velocity and push aside elements having a smaller velocity, in which process usually surfaces of intensive shearing motion are generated. Shearing motion is not represented by the solutions of Eq. (1), but the analogy is retained when we do not look at the shear itself, but at the dissipation accompanying it. A similar dissipation is to be found in each steep front developing in the course of v.

Surfaces of shear flow, separating masses of liquid with different velocities, can merge together in a similar way as the steep fronts in the curve describing our function v. Both in the case of fluid motion and in that case of Eq. (1) this process has the importance that it leads to an increase of the scale of the pattern.

A system intermediate in its character between the complete hydrodynamic equations and Eq. (1) is formed by Eqs. (13) considered shortly in Section 45 of the preceding chapter. The solutions of the system (13) actually exhibit the formation of regions of shear flow, which already are quite comparable to those of actual fluid flow.

It is possible to make a Fourier analysis of the solutions of Eq. (1), in a similar way as this can be done in hydrodynamics. It is then found, in both cases, that there is a coupling between the various components, depending upon the nonlinear terms of the equations, of such nature that the interaction between any two components leads to the appearance, or the influencing, of other components having the sum or the difference of the frequencies. Hence, when we describe the system by means of a spectrum, there is a transfer of energy both upward and downward along the frequency scale. Viscosity in both cases operates most effectively on the components of small wavelength (high frequency), and hence in both cases there must be an overall flow of energy towards the high frequency end of the spectrum. In all these aspects and in a number of statistical features, the solutions of Eq. (1) can help to understand relations appearing in actual turbulence. This will be shown in greater detail in the next sections.

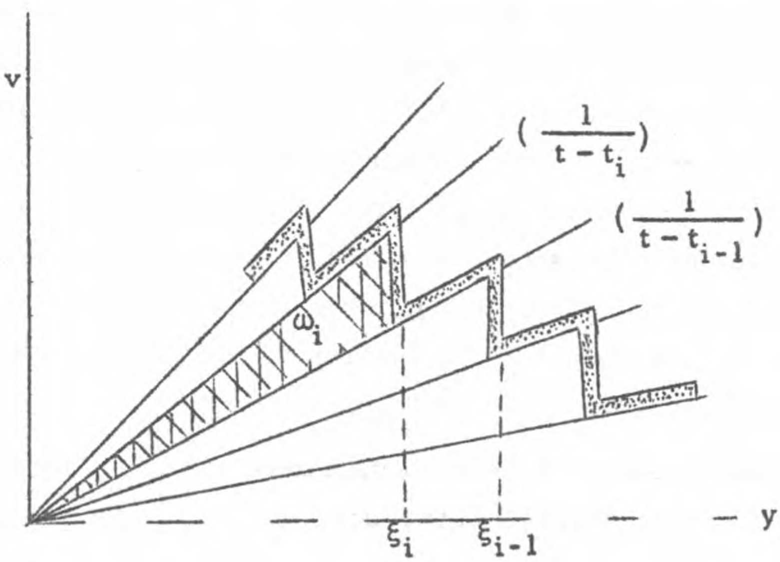
Attention may be drawn to one further point. Although the resolution of a pattern of fluid motion into Fourier components may be convenient from many points of view, there is also something unnatural in it and the strong coupling between the various components due to the nonlinear terms of the equations present a mathematical problem which thus far cannot be solved. It is difficult, therefore, to predict the history of any single Fourier component. The same applies to any other method of resolution based on the characteristic solutions of some linear partial differential equation (for instance, the equation used in investigations on the stability of laminar motion). On the other hand, the type of solutions of Eq. (1) considered in the preceding section, with their steep fronts, are directly connected with the nonlinear term and present, so to say, an "individualized" character. Steep fronts have a certain lifetime and it is possible to describe a certain class of solutions of (1) by giving a list of the positions and strengths of the front. The history of the field then becomes a description of the motion of these fronts and their merging together, leading to a gradual decrease of their number and increase of their mean distance. Such a description brings into evidence those properties of the field which are dependent on the presence of the nonlinear term in the equation, whereas the Fourier resolution is based primarily on the linear terms and is only poorly adapted to the treatment of nonlinear effects.

In the next sections we shall consider various particular cases of fields described by Eq. (1).

54. Solutions of Equation (1) Representing the Propagation of Impulses in One Direction Along the Y-Axis.

In Section 51 we have become acquainted with a particular solution of Eq. (1) corresponding to an impulse of magnitude $h^2/2$ introduced at the instant t_o at the point $y = \delta$. The course of v is represented by a triangle, having the area $h^2/2$, while the velocity of advance of the front is given by one-half the height of the front. These two rules completely determine the solution.

We now imagine that a series of impulses of magnitudes $\omega_1, \omega_2, \omega_3, \ldots$ is introduced at the point $y = 0$ at instants $t_1, t_2, t_3, \ldots$, all impulses being positive. Each impulse leads to the appearance of a triangle and the various triangles are superposed on each other in the way indicated in the accompanying figure. For any instant, this figure can be constructed by drawing a set of straight lines starting from $y = 0$ and having slopes given by $1/(t - t_i)$; between these lines we draw vertical segments in such a way that a triangle with the area ω_i is formed between the lines with slopes $1/(t - t_i)$ and $1/(t - t_{i-1})$. We denote the position of the corresponding vertical segment by ξ_i and provisionally assume that all ξ_i satisfy the conditions $\xi_i < \xi_{i-1}$.



We then have a representation of the course of v at the instant t (heavy broken line in the diagram). The development in time of the curve is determined by two rules: all inclined lines turn to the right in conformity with the expression $1/(t - t_i)$ for their slope; every triangle must retain a constant area ω_i. One finds:

$$\xi_i = \sqrt{\frac{2\,\omega_i(t - t_i)(t - t_{i-1})}{t_i - t_{i-1}}} \tag{9}$$

We assume that new impulses are introduced at $y = 0$ as time advances, which will give rise to new triangles to be superposed at the left end of the curve. The corrections connected with viscosity are neglected in this picture, but we can assume that they are very small.

The curve obtained gives a crude analogy to turbulence produced by a screen in a wind tunnel and carried along by the general flow. In the present case there is no special "carrying flow". The propagation along the y-axis is determined by the magnitude of the impulses themselves. We suppose that at $y = 0$ impulses are continuously introduced in a random manner, but so that it is possible to define a mean value $\overline{\omega}$ of the ω_i and a mean value θ of the time intervals $T_i = t_i - t_{i-1}$, and also a mean value of the ratio ω_i/T_i. The picture then obtained has a stationary statistical character for every given point of the y-axis; it changes gradually as we go along the y-axis.

The supposition that all ξ_i satisfy $\xi_i < \xi_{i-1}$ cannot be upheld in general. The velocities of advance of the various vertical fronts will be unequal and whenever we find $d\,\xi_i/dt > d\,\xi_{i-1}/dt$, there will be a decrease of the distance $\xi_{i-1} - \xi_i$. When this has become zero, the vertical front ξ_i has overtaken the segment ξ_{i-1}; from then onward we must count these fronts as a single one. This requires that from now on we leave out the inclined line with the slope $1/(t - t_{i-1})$ and work as if the combined impulse $\omega_i + \omega_{i-1}$ had been introduced at the instant t_i, so that from now on it follows the impulse ω_{i-2} introduced at t_{i-2}.

This process leads to a gradual decrease of the number of fronts. Since new fronts are continually introduced at $y = 0$, the statistical state of the system can be stationary.

We now come back to the rule for constructing the state of the system at a given instant t. Whenever, in carrying out the rules given before, we find a case where $\xi_i > \xi_{i-1}$, we take out the instant t_{i-1} and combine ω_i and ω_{i-1} into a single impulse corresponding to the instant t_i. If we should find that the new value of ξ_i would surpass ξ_{i-2}, or that it would be smaller than ξ_{i+1}, the process has to be repeated until all cases which did not satisfy the condition $\xi_i < \xi_{i-1}$ have been eliminated. The result of this elimination is not dependent on the order in which it is carried out.

A particular case is obtained when all impulses are exactly of the same magnitude $\overline{\omega}$ and are spaced at exactly equal intervals of time θ. In that case, merging of fronts will occur only at the foremost end of the sequence. If the process has been going on for a long time, this end will have moved far out to the right, and in certain cases can be considered as having disappeared from the field under observation. The velocity of advance of all other vertical fronts will then be very nearly equal to $\overline{\omega}/\theta$. It should be observed that when the initial impulses were not exactly equal, or when there should have been slight inequalities of their spacing in time, these inequalities would not disappear in the process but would lead to the merging of fronts somewhere inside the series. This is a typical feature characteristic for systems which are governed by a nonlinear equation, belonging to the hyperbolic type (having real characteristics), where there is no smoothing out of irregularities introduced by the boundary conditions or by the initial conditions.

In the beginning of this chapter it had been remarked that Eq. (1) is also illustrative of phenomena peculiar to compressible media. The solution we have been considering can be taken as a simplified picture of the behavior of a series of plane shock waves, introduced one after the other into a gas. Real shock waves will show the same feature of overtaking one another and of merging together, as do the steep fronts found in our solution of Eq. (1).

55. Statistical Problems Connected with the Propagation of Consecutive Impulses. If t is chosen so that for a group of consecutive impulses $t - t_i$ is large compared with the time intervals T_i within this group, we can replace (9) by the approximation:

$$(10) \qquad \xi_i \cong u_i(t - t_i + \tfrac{1}{2} T_i)$$

where $u_i = \sqrt{2\,\omega_i/T_i}$.

In this case the condition $\xi_i < \xi_{i-1}$ is not compatible with a large positive difference $u_i - u_{i-1}$. Whenever $u_i > u_{i-1}$ we can find a value of t for which the condition $\xi_i < \xi_{i-1}$ will be violated, so that we must combine impulses in the way described before. Since we have assumed that the system has a statistically stationary character, we can expect that in the course of time all appreciable positive differences between consecutive u_i will be eliminated in this way. There might remain large negative differences $u_i - u_{i-1}$ or series with consecutive negative differences, but, in view of the randomness of the values of the u_i, a large value of u_{i-k} would have great chance to be followed by a smaller one, and then elimination again would be necessary.

To make these considerations more precise, we observe that when two vertical fronts merge together into a single one, the values of the ω_i are added, so as also are the values of the intervals T_i. The new value of u_i resulting from the process is given by:

$$u_i^* = \sqrt{\frac{u_i^2 T_i + u_{i-1}^2 T_{i-1}}{T_i + T_{i-1}}}$$

If now we define a mean square value of the u_i by means of the formula:

$$U^2 = \frac{\Sigma u_i^2 T_i}{\Sigma T_i}, \tag{11}$$

where the summation is extended over a series of consecutive i-values, it follows that this mean square value is not affected by the process of merging together of fronts and that it is a constant. It is easily seen that this constant has the value $2\overline{\omega}/\theta$.

On the other hand, since

$$u_i^* T_i^* = \sqrt{(u_i T_i + u_{i-1} T_{i-1})^2 + T_i T_{i-1} (u_i - u_{i-1})^2}, \tag{12}$$

where $T_i^* = T_i + T_{i-1}$; it is seen that $\Sigma u_i T_i$ increases each time two fronts merge. Hence the (linear) mean value $\overline{u}$ of the u_i increases. It thus follows that the fluctuations of the u_i tend to become smaller and smaller as time goes on.

This can be put into a slightly different form if we write:

$$u_i = U(1 + \delta_i). \tag{13}$$

We then have for the mean value of the square of $1 + \delta_i$:

$$\overline{(1 + \delta_i)^2} = 1 ,$$

from which:

$$2\overline{\delta} = - \overline{\delta^2} .$$

Hence $\overline{\delta} < 0$ and $\overline{u} < U$. Since $\overline{u}$ increases, $\overline{\delta}$ increases towards zero and $\overline{\delta^2}$ decreases. This implies that the absolute values of the δ_i gradually become smaller.

A consequence of these considerations is that, if we have started with two independent distributions for the values of the ω_i and the T_i, these will not remain independent of each other. If they are represented simultaneously by means of points in an ω_i, T_i diagram, the combinations occurring every time when two fronts merge will lead to the appearance of both larger ω_i values and larger T_i values in such a way that the ratios ω_i/T_i will cluster more and more closely around the mean value $\overline{\omega}/\theta$.

56. Application of Similarity Considerations. We will follow the history of a group, originally formed from N_0 impulses of total amount Ω, introduced at $y = 0$ in a period of duration Θ. When we consider the group at an instant t, the number of fronts will have decreased to say N, and the mean values $\overline{\omega}$ and θ are given by: $\overline{\omega} = \Omega/N$; $\theta = \Theta/N$.

When the merging process has already gone so far that the values of the δ_i are small compared with unity, it follows from (12) that when two fronts merge (which makes N decrease by 1) the sum $\Sigma u_i T_i$ increases by approximately:

$$\frac{T_i T_{i-1}(u_i - u_{i-1})^2}{2(u_i T_i + u_{i-1} T_{i-1})} .$$

It seems fairly safe to assume that the mean value of this quantity is given by:

$$\frac{1}{4} U \theta \overline{(\delta_i - \delta_{i-1})^2} = \frac{1}{2} U \theta \overline{\delta^2} .$$

Hence the mean value $\overline{u}$ increases on the average by $\frac{1}{2} U \overline{\delta^2}/N$, from which there further follows:

$$\text{average increase of } \overline{\delta} = \frac{1}{2} \overline{\delta^2}/N$$

and:

$$\text{average decrease of } \overline{\delta^2} = \overline{\delta^2}/N .$$

If we assume that a certain statistical similarity is conserved in the process, we must write this in the form:

$$\frac{\text{decrease of } \overline{\delta^2}}{\overline{\delta^2}} = \frac{\text{decrease of } N}{N} ,$$

and find:

$$(14) \qquad \overline{\delta^2} \backsim N .$$

Now the instant t^* at which ξ_i and ξ_{i-1} merge, (this was the typical combination considered in the formulas, assuming that $u_i - u_{i-1} > 0$) is given by:

$$t^* - t_i + \frac{1}{2} T_i \cong \frac{u_{i-1}(T_i + T_{i-1})}{2(u_i - u_{i-1})} \cong \frac{(1+\delta_i)(T_i + T_{i-1})}{2(\delta_i - \delta_{i-1})} .$$

We can take this expression as an estimate of the "age" of the group under consideration. Since the average value of $| \delta_i - \delta_{i-1} |$ is $\sqrt{2\overline{\delta^2}}$, we obtain:

$$\text{average age} \cong \theta / \sqrt{2\overline{\delta^2}} \backsim \Theta N^{-3/2} .$$

In this way we arrive at the results:

$$(15) \qquad N \backsim (\text{age})^{-2/3}$$

$$(16) \qquad \overline{\delta^2} \backsim (\text{age})^{-2/3} .$$

It also follows that:

$$(17) \qquad \overline{\omega} \text{ and } T \backsim N^{-1} \backsim (\text{age})^{2/3} .$$

The dependence on the age can be transformed into dependence on the mean position of the group along the y-axis, if we observe that according to (10):

$$(18) \qquad \xi_i \backsimeq U \cdot (\text{age}) \ .$$

57. An observer located at a fixed value of y will see the broken curve representing the course of v pass over him with a velocity approximately equal to U. His observations will show a slow increase of v during periods T_i, alternating with a sudden decrease. The magnitude of the decrease, observed at the instant when the front ξ_i passes the observer, is given by:

$$(19) \qquad \left[v_i\right] = \sqrt{\frac{2\omega_i T_i}{(t-t_i)(t-t_{i-1})}} \backsimeq \frac{\sqrt{2\omega_i T_i}}{t-t_i+\frac{1}{2}T_i} \ .$$

From this expression it appears that so long as merging of fronts does not take place, the values of v_i (which are the quantities $v_i - v_{II}$ considered in Section 52) decrease inversely proportionally with age, and consequently also inversely proportionally with the distance of the observer from the origin on the y-axis. However, every merging of two successive fronts introduces an increase of ω_i and T_i; the decrease of the average value of the v_i consequently will be slower. From (17) we see that the numerator increases proportionally with $(\text{age})^{2/3}$. Hence we find:

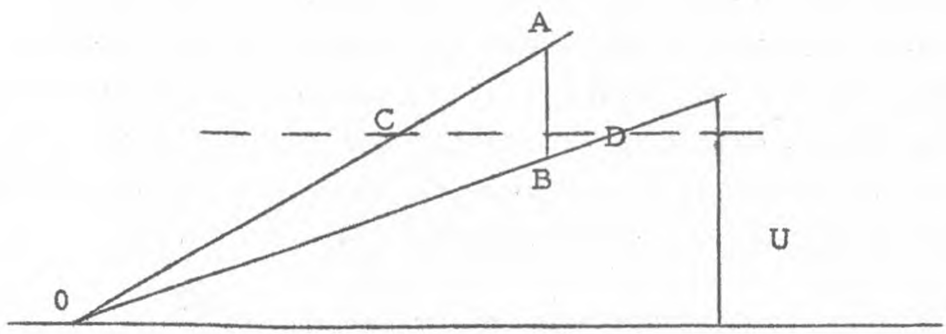
$$(20) \qquad \text{average value of } \left[v_i\right] \backsim (\text{age})^{-1/3}.$$

Certain other problems connected with the v-curve passing over a stationary observer can be considered. Since the process should be stationary in the statistical sense, we can consider the Eulerian time-correlation $\overline{v(t)\,v(t+\tau)}$ for a fixed value of y. It can be estimated that this correlation will become zero when τ exceeds a few times θ.

We can also fix attention to a particular point of the curve for v. Such a point displaces itself without change of height with its velocity v, until the instant where the two fronts between which it is situated happen to merge. The interval of time, from the instant at which the corresponding impulses were introduced until the instant at which the fronts merge, is of the order of the age of the group to which the fronts belong. Since this age is much larger than θ, an individual point of the curve, followed in its motion, will usually keep its velocity over a long period. Hence a Lagrangian correlation calculated for the motion of individual points will be different from zero over a much longer interval than the Eulerian time correlation.

When the age of a group is large, and the group still contains many members, it is also possible to calculate a Eulerian space correlation $\overline{v(y)\, v(y+\eta)}$ for the group, which quantity will be a function of time. These Eulerian space correlations, however, can be better investigated if we turn to a different type of solution of Eq. (1) in which the broken curve giving the course of v is formed of parallel upward sloping segments (instead of segments all starting from the point $y = 0$). Such a solution can be obtained from the one considered here by means of a limiting process, but it is more convenient to define it in an independent way.

It is useful to notice the quantity $\omega_i - \frac{1}{2} U^2 T_i$ for the solution thus far considered. This quantity represents the difference



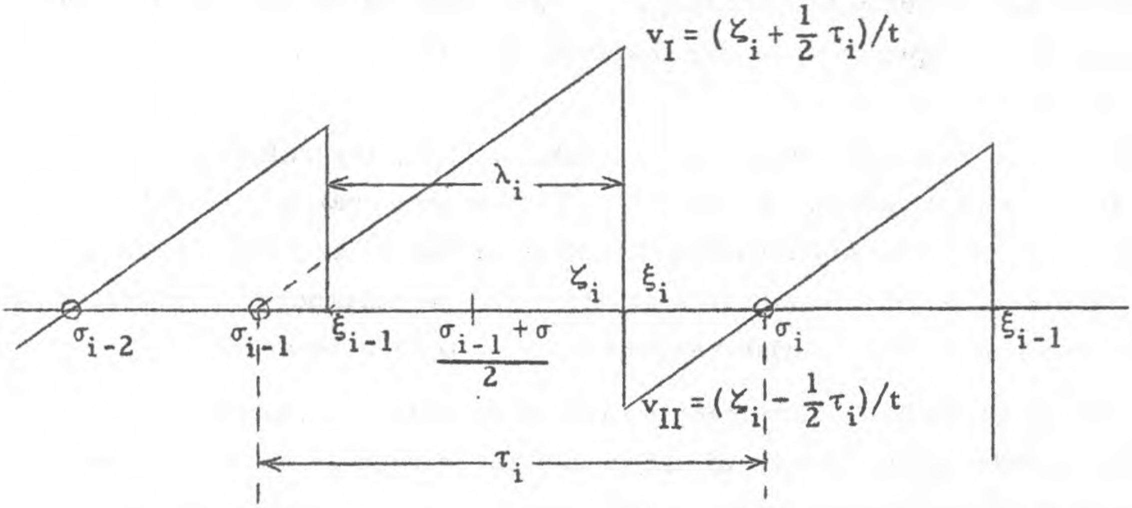
in area between the two triangles 0AB and 0CD connected with the interval $T_i = t_i - t_{i-1}$. The value can be positive or negative and the mean value is zero. Since

$$\omega_i - \frac{1}{2} U^2 T_i = U^2 T_i (\delta_i + \frac{1}{2}\delta_i^2) \cong U^2 T_i \delta_i ,$$

we must expect that the average absolute value will increase proportionally with $(\text{age})^{1/3}$.

Solutions of Eq. (1) Illustrating Spatially Homogeneous Turbulence Decaying with Time

58. We consider a solution of Eq. (1) in which the initial form of the curve for v is given by a series of straight parallel segments. In consequence of formulas (2) and (3) of Section 50, the parallelism will be retained during the whole history, though the magnitude of the slope will decrease. It is convenient to take the slope equal to $1/t$, omitting the unimportant constant t_o. The upward sloping segments are again separated by vertical fronts. If account must be taken of the influence of viscosity, a more correct expression for the course of v in these fronts can be given by making use of Eq. (8) of Section 52. If we introduce the notation which is illustrated in the accompanying diagram, it will be seen that the height of a vertical front is measured by $v_I - v_{II} = \tau_i/t$. We change the first term of (8) in such a way



that account is taken of the slope of the curve to the left and to the right of the front and obtain:

$$(21) \qquad v = \frac{y - \frac{1}{2}(\sigma_i + \sigma_{i-1})}{t} - \frac{\tau_i}{2t} \tanh \frac{\tau_i (y - \xi_i)}{4\nu t} .$$

At the same time:

$$(22) \qquad \frac{d\xi_i}{dt} = \frac{\xi_i - \sigma_i + \frac{1}{2}\tau_i}{t} .$$

The distribution of the lengths τ_i and λ_i to a large extent can be chosen arbitrarily at the initial instant, provided the distribution is sufficiently homogeneous in order that mean values $\overline{\overline{\tau}}$ and $\overline{\overline{\lambda}}$ shall exist. These mean values are obtained by taking a certain number N of consecutive τ_i of λ_i and dividing their sum by N. The result must approach a definite limit when N increases more and more, which limit should be independent of the choice of the starting point. The mean values $\overline{\overline{\tau}}$ and $\overline{\overline{\lambda}}$ moreover shall be equal. In consequence of the circumstance that the development of the system in the course of time leads to the merging of fronts, there can arise a certain interdependence between the distributions of the τ_i and λ_i, and we shall see later (Section 65) that such an interdependence may be of importance.

In the calculations we shall also have to do with mean values obtained by integrating a quantity with respect to y over a certain length S of the y-axis and dividing by this length. Such mean values will be denoted by means of a simple bar, as for instance $\overline{y}$. It is often convenient to make $S = N\ell$.

There is still a freedom in the system, i.e., the relative situation of the two series of points σ_i and ξ_i. We assume that this is determined in such a way that the statistical properties of the system do not change when the direction of y and the sign of v are simultaneously changed. The mean value $\overline{v}$ of v with respect to y will then be zero.

The properties of homogeneity and of statistical invariance with respect to simultaneous change of order and of the sign of v, just mentioned, remain valid throughout the development of the system if they have existed at some initial instant. This is a consequence of the invariance of the differential equation, both with respect to a shift along the y-axis and with respect to a simultaneous change of signs of v and y.

Concerning the development of the system in the course of time, we observe that the σ_i and the lengths τ_i are constants. The points ξ_i move according to Eq. (22). It follows that the lengths λ_i are functions of time and

$$\frac{d\lambda_i}{dt} = \frac{\lambda_i - \frac{1}{2}(\tau_i + \tau_{i-1})}{t}$$

It is convenient to introduce quantities ζ_i defined by:

$$(23) \qquad \zeta_i = \xi_i - \frac{1}{2}(\sigma_i + \sigma_{i-1}) = \xi_i - \sigma_i + \frac{1}{2}\tau_i ;$$

then:

$$\frac{d\xi_i}{dt} = \frac{d\zeta_i}{dt} = \frac{\zeta_i}{t} ; \quad \frac{d\lambda_i}{dt} = \frac{\zeta_i - \zeta_{i-1}}{t} .$$

The ζ_i can be positive as well as negative, and the mean value $\overline{\zeta}$ must be zero.

The property of statistical invariance with respect to a simultaneous change of sign of v and y, referred to before, can also be expressed by the rule that any statistical quantity formed from the τ_i, λ_i and ζ_i will remain unchanged when the signs of all the ζ_i are changed simultaneously with a change of the direction in which i is counted, no change of sign being made in τ_i and λ_i. For instance:

$$\overline{\lambda_i \tau_i} = \overline{\lambda_i \tau_{i-1}} ; \quad \overline{\lambda_i \zeta_i} = \overline{-\lambda_i \zeta_{i-1}} ; \quad \overline{\tau_i \zeta_i} = 0 .$$

When two consecutive ξ_i (say ξ_{i-1} and ξ_i) become equal to each other, the point ξ_{i-1} and the segment λ_i disappear from the arrangement. The values of τ_{i-1} and τ_i are added and the law of motion of the resulting segment is again determined by (22), provided we replace τ_i by $\tau_i + \tau_{i-1}$. By this process both the number of segments τ_i and the number of segments λ_i are gradually reduced in the course of time.

59. <u>Simple Mean Values.</u> We write:

$$(24a) \qquad \overline{\tau} = \overline{\lambda} = \ell$$

and further:

$$(24b) \qquad \overline{\tau^2} = \ell^2(1+\omega) ; \quad \overline{\tau^3} = \ell^3(1+\omega^*) ; \quad \overline{\tau_i \zeta_i^{\,2}} = \ell^3 \overline{\omega} .$$

To calculate the mean value of v (which is zero, as mentioned before), we observe that v changes linearly with y over a segment λ_i; hence for a single segment:

$$\int v\,dy = t\int v\,dv = \frac{1}{2t}\left[(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^2\right].$$

We sum this expression with respect to i over all segments contained in a part of the y-axis of great length S. The sum must be divided by $S = N\ell$, where N is the number of segments λ_i (and τ_i) contained in S. From $\overline{\tau_i\,\zeta_i} = 0$, it immediately follows that $\overline{v} = 0$.

To find the mean value of v^2 a similar procedure is applied. Integration over the length of the segment λ_i gives:

$$\int v^2\,dy = t\int v^2\,dv = \frac{1}{3t^2}\left[(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^3\right].$$

Hence the mean value becomes:

$$(25)\qquad \overline{v^2} = \frac{1}{\ell t^2}\left(\overline{\tau_i\zeta_i^2} + \frac{1}{12}\overline{\tau^3}\right) = \frac{\ell^2}{t^2}\left[\overline{\omega} + \frac{1}{12}(1+\omega^*)\right].$$

It has been mentioned in Section 52 that the dissipation is to be found only in the vertical segments, each segment giving a contribution which in the present notation is equal to $\frac{1}{12}\tau_i^3/t^3$. It follows that the mean dissipation of energy in unit time per unit length of the y-axis is given by:

$$(26)\qquad \epsilon = \frac{1}{12\,\ell t^3}\,\overline{\tau^3} = \frac{\ell^2}{12\,t^3}(1+\omega^*).$$

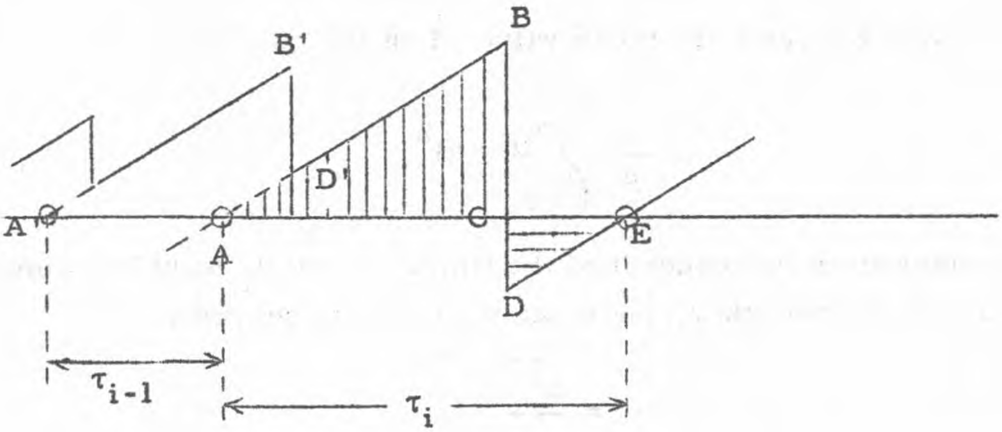
In consequence of the merging of fronts the number of segments N contained in a constant great length S decreases and ℓ increases with time. It is also possible that the values of the dimensionless parameters ω, ω^*, $\overline{\omega}$, will be functions of t. If the arrangement retains a statistical similarity, these parameters will be constants.

So long as we do not know how ℓ changes with time, very little can be said about the behavior of v^2 and ϵ.

60. Quantities Connected with the Momentum Integral. We write:

$$(27)\qquad \mu_i = \tau_i\,\zeta_i/t\,.$$

Every quantity μ_i corresponds to a segment τ_i and to a vertical front with position ξ_i. With the aid of the relations indicated in the diagram of Section 58, it can easily be proved that μ_i is the algebraic value of the area ABCDE



contained between the two sloping lines connected with the segment τ_i (counting ABC positive and CDE negative). For the segment τ_{i-1} in the diagram given above, where the situation is somewhat different, the corresponding quantity would be the area A'B'C'A, counted positive. The quantities μ_i, which thus can be positive as well as negative, are the analogues of the quantities $\omega_i - \frac{1}{2} U^2 T_i$ considered at the end of Section 57.

It will be seen that the integral

$$\int_y^{y+S} v \, dy$$

extended over a certain length S of the y-axis will be equal to

$$\sum_{k=0}^{N-1} \mu_{i+k} + \Delta ,$$

where the sum refers to the values of μ_i for all segments τ_i contained in S, while Δ stands for the additional parts appearing at the ends of the integration interval.

Since $d\zeta_i/dt = \zeta_i/t$, from which it follows that the ζ_i are proportional with t, the μ_i are independent of the time. Whenever two ξ_i come to coincidence and the corresponding vertical fronts merge into a single front, the μ_i corresponding to these fronts are added. Hence $\Sigma \mu_i$ does not change. This result is evidently connected with the property of the momentum integral mentioned in Section 52.

We now consider the mean value M of the quantity:

$$\frac{1}{S}\Big(\sum_{k=0}^{N-1} \mu_{i+k}\Big)^2$$

which evidently is independent of the time. If the μ_i would be completely independent of each other, the mean value would reduce to:

$$M = \frac{\overline{\mu^2}}{\ell}$$

It is possible, however, that relations exist between successive μ_i, so that the mean values $\overline{\mu_i \, \mu_{i+k}}$ may be different from zero. We can expect nevertheless that these mean values rapidly will approach zero when k increases. Hence, when N is large, we may write:

$$(28) \qquad M = \frac{1}{\ell}\left[\overline{\mu^2} + 2 \sum_{k=1}^{\infty} \overline{\mu_i \mu_{i+k}}\right]$$

which formula contains the preceding one as a special case.

> If the arrangement should retain a statistical similarity (which in itself is not quite certain, we shall see afterwards), an important argument could be deduced from the fact that M is independent of the time. If there would be statistical similarity, mean values like $\overline{\mu^2}$ and $\overline{\mu_i \mu_{i+k}}$ would be proportional to the fourth power of the mean length ℓ of the segments τ_i, divided by the square of the time t. Hence M would become proportional to the third power of ℓ divided by t^2, and since this must be a constant, it follows that ℓ must increase according to the formula:
>
> $$(29) \qquad \ell \backsim t^{2/3}$$
>
> As ω, ω^*, $\overline{\omega}$ would be constants, it further follows from Eqs. (25) and (26) that:
>
> $$(30) \qquad v^2 \backsim t^{-2/3}; \quad \epsilon \backsim t^{-5/3}.$$

These results would be in accordance with those obtained in Sections 56 and 57 for the preceding type of solutions (see Eqs. 17 and 20).

Even if there should be similarity, the argument would break down if M would be zero. This can occur where there exist certain relations between the mean values $\overline{\mu_i \mu_{i+k}}$ and $\overline{\mu^2}$, of such nature that the mean value of $(\Sigma\mu_{i+k})^2$ would not increase proportionally with N, but at a slower rate or would approach to a constant value.

With reference to this point, it is of importance to mention that the solutions we have been considering can be obtained from an initial state in which a series of concentrated impulses, each of a finite integrated magnitude A_m, is introduced at an infinite series of points of the y-axis, arbitrarily spaced, but so that the distribution is statistically homogeneous. The A_m can be positive or negative and follow each other at random, while the mean value of the A_m over any large domain of the y-axis shall be zero. It can then be proved that the μ_i are either equal to certain A_m or are equal to sums of consecutive A_m (in consequence of the merging together of consecutive impulses), so that $\Sigma\mu_{i+k}$ over any length S of the y-axis is equal to ΣA_m for the same length. Hence if the random distribution of the A_m is subjected to the condition that the mean value of $(\Sigma A_m)^2$ will not increase proportionally with the length S, but increases at a slower rate or approaches to a constant value, the same property will apply to $(\Sigma\mu_{i+k})^2$. Such arrangements can be obtained by choosing a particular rule for determining the magnitudes of the A_m.

In the next section we shall see that M is connected with an important invariant referring to correlation functions.

61. Correlations. For the type of solutions we are considering now, the Eulerian space correlation $\overline{v(y)\, v(y+\eta)}$ can be obtained, for which we shall write $\overline{v_1 v_2}$. Another important Eulerian correlation function is $\overline{[v(y)]^2\, v(y+\eta)}$, to be written $\overline{v_1^2 v_2}$. It will be evident that $\overline{v_1 v_2}$ is a symmetric function of η, having its maximum at $\eta = 0$, whereas $\overline{v_1^2 v_2}$ is an odd function of η, which is zero for $\eta = 0$. We have $\overline{v_1^2 v_2} = -\overline{v_1 v_2^2}$. The function $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ both decrease to zero when η becomes large. It is understood that the mean values refer to a particular instant. Both quantities, therefore, are functions of the time.

Starting from Eq. (1), the following equation can be formed [where $v_1 = v(y)$; $v_2 = v(y+\eta)$] :

$$\frac{\partial}{\partial t}(v_1 v_2) = -v_1 v_2 \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial y} \right) + \nu \left(v_2 \frac{\partial^2 v_1}{\partial y^2} + v_1 \frac{\partial^2 v_2}{\partial y^2} \right).$$

When mean values are taken from both sides, we obtain (according to a well known procedure):

$$(31) \qquad \frac{\partial}{\partial t} \overline{(v_1 v_2)} = \frac{\partial}{\partial \eta} \overline{(v_1^2 v_2)} + 2\nu \frac{\partial^2}{\partial \eta^2} \overline{(v_1 v_2)} .$$

This equation is the analogue of the important equation of von Karman and Howarth for hydrodynamic turbulence. It has an important place in investigations on the statistical behavior of the solutions we are considering. If we consider not too small values of η, the term with ν can be discarded.

Integrating (31) with respect to η from 0 to ∞ we find:

$$(32) \qquad dJ/dt = 0$$

where

$$(32a) \qquad J = \int_0^\infty \overline{v_1 v_2} \, d\eta$$

The quantity J is the analogue of Loitsiansky's invariant in hydrodynamic turbulence.

To obtain the connection between this invariant and the quantity M of the preceding section, we write:

$$M = \frac{1}{S} \overline{\left(\int_y^{y+S} v \, dy \right)^2} .$$

The error made by neglecting the quantity denoted by Δ decreases to zero when S is made larger and larger. The mean value indicated by the bar must be understood in the following way: we calculate the integral with various starting points y, keeping S constant; then we determine the mean value with respect to the starting point. The expression for M can also be written:

$$M = \frac{1}{S}\int_o^S dy_1 \int_o^S dy_2\ \overline{v(y+y_1)\ v(y+y_2)}$$

It will be seen that $\overline{v(y+y_1)\ v(y+y_2)}$ reduces to the function $\overline{v_1 v_2}$ with $\eta = y_1 - y_2$. It is thus easily found that the double integration gives:

$$M = \frac{1}{S}\ .\ 2S \int_o^\infty \overline{v_1 v_2}\ d\eta = 2J\ .$$

62. Scales Connected with the Solutions under Consideration. In all investigations concerning turbulence the concept of certain scales referring either to the macroscopic aspect or to features of detail, plays an important part. Similar quantities can be formed for our solutions of Eq. (1).

A macroscopic scale of length is given by ℓ, representing the mean values $\overline{\tau} = \overline{\lambda}$.

An average amplitude for v is given by ℓ/t. Making use of both quantities we can define a Reynolds number:

(33) $\quad Re = \ell^2/\nu t\ .$

This should be a large number; otherwise the approximations involving either the neglect of the viscosity or, if greater accuracy is needed, the use of the expression (21), would not be valid. If the arrangement in the system would remain statistically similar and the results obtained in Section 60 could be applied, it is found that Re would increase proportionally with $t^{1/3}$.

A microscale m can be defined by means of the development of $\overline{v_1 v_2}$ in powers of η, when this is written in the form:*

(34) $\quad \overline{v_1 v_2} = \overline{v^2}\left(1 - \frac{\eta^2}{2m^2} + \ldots .\right)$

---

*In the existing literature this microscale is denoted by λ . We used m to avoid confusion with the segments λ_i .

According to a well known formula, we then have

$$\overline{\left(\frac{\partial v}{\partial y}\right)^2} = \frac{\overline{v^2}}{m^2} .$$

We introduce the mean kinetic energy $E = \frac{1}{2}\overline{v^2}$ and the mean dissipation $\epsilon = \nu\,\overline{(\partial v/\partial y)^2}$; we then have the equation:

$$\frac{dE}{dt} = -\epsilon\,, \text{ or } \frac{d\overline{v^2}}{dt} = -\frac{2\nu\overline{v^2}}{m^2} . \tag{35}$$

Making use of Eqs. (25) and (26) we obtain:

$$m^2 = \frac{\overline{\tau^3} + 12\,\overline{\tau_i \zeta_i^2}}{\overline{\tau^3}}\,\nu t . \tag{36}$$

(It follows that for a system for which the arrangement in the large would remain statistically similar, we should have:

$$m \backsim \sqrt{\nu t} .$$

This would mean that there could be no similarity over the whole range, since ℓ/m would increase with time.)

One can define a Reynolds number connected with the microscale and obtain:

$$Re_m \backsim Re^{1/2} .$$

63. When we develop the correlation function $\overline{v_1^2 v_2}$ with respect to η, it is found that the term of the first degree disappears, in consequence of the fact that the mean value of $v^2 \cdot (\partial v/\partial y)$ is zero. Since $\overline{v_1^2 v_2}$ is an odd function of η, its development begins with a term in η^3.

The series for the two functions $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ can be written:

$$v_1 v_2 = v^2 - \frac{\eta^2}{2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\eta^4}{24}\left(\frac{\partial^2 v}{\partial y^2}\right)^2 - \dots$$

$$\overline{v_1^2 v_2} = -\frac{\eta^3}{6}\,\overline{\left(\frac{\partial v}{\partial y}\right)^3} + \dots$$

Approximate expressions for the mean values in the right-hand members can be obtained by making use of Eq. (21). The contributions from the upward sloping segments and from the nearly vertical fronts can be calculated separately. The values obtained are:

	upward sloping segments	vertical fronts
$\overline{\frac{\partial v}{\partial y}}$	$\frac{1}{t}$	$-\frac{1}{t}$
$\overline{\left(\frac{\partial v}{\partial y}\right)^2}$	$\frac{1}{t^2}$	$+\frac{\overline{\tau^3}}{12\nu \ell t^3}$
$\overline{\left(\frac{\partial v}{\partial y}\right)^3}$	$\frac{1}{t^3}$	$-\frac{\overline{\tau^5}}{120\nu^2 \ell t^5}$
$\overline{\left(\frac{\partial^2 v}{\partial y^2}\right)^2}$	0	$+\frac{\overline{\tau^5}}{240\,\nu^3 \ell t^5}$

It must be observed, however, that these formulas give the most important parts only, and a more refined calculation would bring terms with smaller powers of ν in the denominators.

It will be seen that the over-all mean value of $\partial v/\partial y$ is zero, as is necessary. The mean values of the other quantities are practically given by the contributions from the fronts. The fact that the mean value of $(\partial v/\partial y)$ is different from zero, is an indication of a certain skewness in the distribution of v. A similar skewness appears in hydrodynamic turbulence. A "skewness factor" can be defined by:

$$\frac{\overline{(\partial v/\partial y)^3}}{\left[\overline{(\partial v/\partial y)^2}\right]^{3/2}} = \frac{\sqrt{12}}{10}\sqrt{\frac{\ell^2}{\nu t}}\;\frac{\overline{\tau^5}}{\left[\overline{\tau^3}\right]^{3/2}\ell^{1/2}} ,$$

which is proportional to $Re^{1/2}$. The latter result is different from what is obtained in hydrodynamic turbulence, where values of order unity are found (Batchelor gives: -0.39; compare Chapter IV, Section 21). The concentration of the gradient of v in the model system, which is not subjected to any equation of continuity, is much larger than the concentration of vorticity in the vortex sheets between eddies in actual turbulence. The series for the correlation functions now takes the form:

$$\overline{v_1 v_2} = \overline{v^2} - \frac{\eta^2}{2}\,\frac{\overline{\tau^3}}{12\nu\,\ell t^3} + \frac{\eta^4}{24}\,\frac{\overline{\tau^5}}{240\nu^3\,\ell t^5} - \cdots\cdots$$

$$\overline{v_1 v_2} = -\frac{\eta^3}{6}\,\frac{\overline{\tau^5}}{120\,\nu^2\,\ell t^5} + \cdots\cdots$$

When these results, which are applicable only for very small values of η, are substituted into (31) and terms independent of η are compared, we obtain:

$$\frac{d\overline{v^2}}{dt} = -\frac{\overline{\tau^3}}{6\,\ell t^3} = -2\,\epsilon\ ,$$

which is the same as Eq. (34) (compare Eq.(26) for ϵ). Comparison of terms in η^2 does not lead to a useful result, since the terms with $\overline{\tau^5}$ should be completed with other terms, for which no expression has been obtained. (In current turbulence theory the comparison of the terms in η^2 is used for discussing the decay of vorticity, but also here in a qualitative way only, since detailed expressions for the quantities appearing in the equations are unknown.)

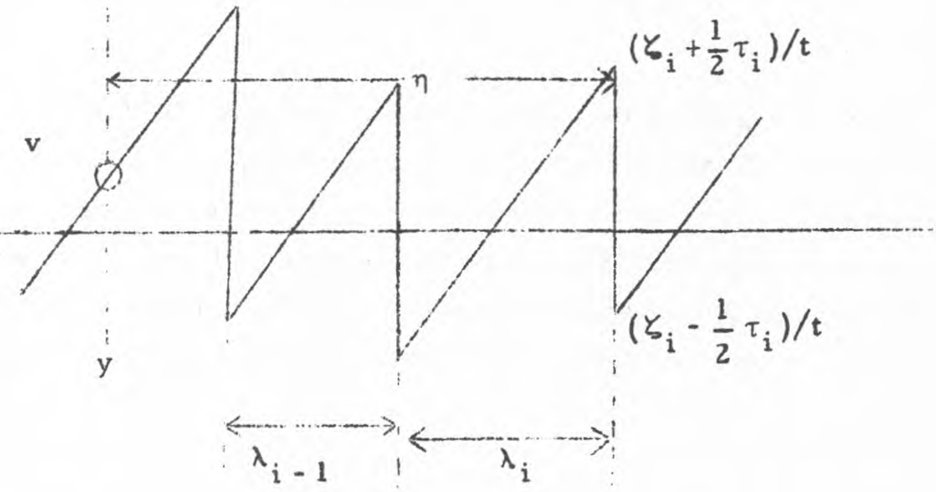
Expressions for the Correlation Functions $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$

64. For further discussion of the statistical properties of the solutions it is necessary to obtain expressions for the correlation functions $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ and their first derivatives with respect to η. This will also bring us to certain conditions to be satisfied by the arrangement of the segments τ_i and λ_i. Since in constructing the formulas it is necessary to distinguish between various cases, the full mathematical deduction is rather involved. The most simple case is presented by $\partial(v_1 v_2)/\partial\eta$ and if we content ourselves with a less rigorous deduction, the expression for this quantity can be obtained in the following way.

We start the relation:

$$\partial\,\overline{(v_1 v_2)} = \overline{v(y)\frac{\partial}{\partial y}\,v(y+\eta)} = \overline{v(y-\eta\,\frac{\partial v}{\partial y})}$$

Along the upward sloping segments of the curve for v the derivative $\partial v/\partial y$ has the constant value $1/t$. It follows that the contribution of these segments to the desired mean value is given by $\overline{v}/t$, which is zero. In the nearly vertical fronts $\partial v/\partial y$ assumes large negative values, the integrated amount for any particular front being equal to $-\tau_i/t$. The value of v to be multiplied with this quantity is given by



$(\zeta_i + \frac{1}{2}\tau_i - \eta)/t$ if $\eta \quad \lambda_i$; or more generally by $(\zeta_i + \frac{1}{2}\tau_i + \tau_{i-1} + \dots + \tau_{i-k} - \eta)/t$ if η should exceed the combined length of the k segments $\lambda_i + \lambda_{i-1} + \dots + \lambda_{i-k+1}$, where k can be any number from 1 upward. Since all these cases can occur, the mean value of $v(y-\eta)\cdot(\partial v/\partial y)$ is obtained as the mean with respect to y of the expression:

$$-\frac{\tau_i}{t^2}(\zeta_i + \frac{1}{2}\tau_i - \eta + g_1\tau_{i-1} + g_2\tau_{i-2} + \dots),$$

where the coefficients g_k (which are functions of η) specify the probability that η will exceed the sum of k consecutive segments. We first take the mean value with respect to i and afterwards divide by ℓ in order to obtain the mean with respect to y. The mean value of τ_i is ℓ; that of $\tau_i\zeta_i$ is zero; and we find:

$$\text{(37)} \qquad \frac{\partial(\overline{v_1 v_2})}{\partial\eta} = -\frac{\overline{\tau^2}}{2\ell t^2} + \frac{\eta}{t^2} - \frac{1}{\ell t^2}\sum_{k=1}^{\infty} g_k \, \overline{\tau_i\tau_{i+k}}$$

To obtain the g_k we introduce a set of distribution functions f_k. The first one, f_1, of these functions gives the distribution of the possible values of the length λ_i of a single segment. The second one, f_2, gives the distribution of the combined length of two consecutive segments, say $\lambda_{i+1} + \lambda_{i+2}$; and f_k gives the distribution of the combined length of k consecutive segments. The functions g_k are the integrals of the f_k:

$$g_k(\eta) = \int_0^\eta f_k(\lambda)\, d\lambda .$$

A more rigorous deduction, paying attention to details, is given in Proc. Netherl. Acad. Sciences, Amsterdam 53, p. 393, 1950. There the expression for $\overline{v_1 v_2}$ is deduced first, in order to eliminate difficulties concerning derivatives; the expressions obtained have been written in such a form that it is possible to take account of correlations which might exist between the values of $\overline{\tau_i \tau_{i+k}}$ and the distances $\xi_{i+k} - \xi_i = \lambda_{i+1} + \lambda_{i+2} + \ldots \lambda_{i+k}$.

65. We must expect that all correlation functions and thus also (37) will become zero for infinite values of η. A verification of this property reveals that certain conditions must be fulfilled by the arrangement of the τ_i and λ_i. We write:

$$\overline{\tau^2} = \ell^2(1+\omega); \quad \overline{\tau_i \tau_{i+k}} = \ell^2(1+\omega_k) .$$

The ω_k (with k different from zero) will be zero if there are no correlations between consecutive τ_i. They may be different from zero if there exist such relations, but nevertheless we can expect that the ω_k will vanish for large k. Since for a large value of η all g_k with k small compared with η/ℓ will be practically equal to unity, we find:

$$\sum g_k \overline{\tau_i \tau_{i+k}} = \ell^2 \sum g_k + \ell^2 \sum \omega_k .$$

The second sum is independent of η. Concerning the first sum, we observe that we can assume:

$$\sum_1^\infty f_k(\lambda) = 1/\ell \text{ for large } \lambda .$$

Indeed, $\Sigma f_k(\lambda)\, d\lambda$ is the probability to find a vertical front at a point ξ_{i+k} with arbitrary k, satisfying $\xi_i + \lambda < \xi_{i+k} < \xi_i + \lambda + d\lambda$. This probability must become independent of λ when λ is large enough, provided there is sufficient randomness in the arrangement of the λ_i. It will then be equal to $d\lambda/\ell$. It follows that:

$$\sum g_k = \eta/\ell + \text{constant.}$$

This proves that (37) becomes independent of η for large η. It does not prove that (37) will become zero; for this it is necessary that:

$$\text{(38)} \qquad \sum_1^{\infty} g_k(\eta) = \frac{\eta}{\ell} - \frac{1}{2} - \frac{\omega}{2} - \sum \omega_k .$$

The condition (38) is not fulfulled when we take all τ_i and λ_i to be completely independent of each other and to be subjected to no other condition than that the mean values $\overline{\tau}$ and $\overline{\lambda}$ shall be equal to ℓ. For instance, if we take the simple distribution function:

$$f_1(\lambda)\, d\lambda = e^{-\lambda/\ell}\, d\lambda/\ell ,$$

which, with complete independence of the λ_i, leads to:

$$f_k(\lambda) = \frac{1}{(k-1)!}\left(\frac{\lambda}{\ell}\right)^{k-1} e^{-\lambda/\ell}\, \frac{d\lambda}{\ell} ,$$

we find:

$$\sum f_k = 1/\ell \quad \text{for all} \quad \eta ;$$

and

$$\sum g_k = \eta/\ell \quad \text{for all} \quad \eta .$$

Furthermore, complete mutual independence of the τ_i gives $\omega_k = 0$ for all k different from zero. Equation (37) in this case becomes:

$$\frac{\partial\,\overline{(v_1 v_2)}}{\partial\,\eta} = -\frac{\overline{\tau^2}}{2\ell t^2} \quad \text{for all} \quad \eta .$$

The result arrived at is understandable if we observe that with completely independent τ_i and λ_i the curve for v becomes of a type as is found in problems of the "random walk", with arbitrary positive steps (λ_i/t) alternating with arbitrary negative steps (τ_i/t). It is known that in such a case the mean square difference $\overline{(v_1 - v_2)^2}$ between the values of v at two points a large distance η apart, increases linearly with η. Indeed, we roughly have:

$$v_1 - v_2 \cong \frac{1}{t} \sum_1^N (\tau_{i+k} - \lambda_{i+k})$$

where N = largest integer in η/ℓ; and with complete independence:

$$\overline{(v_1 - v_2)^2} \cong \frac{1}{t^2} \overline{\left[\sum (\tau_{i+k} - \lambda_{i+k}) \right]^2} \cong \frac{N}{t^2} \overline{(\tau_i - \lambda_i)^2} =$$

$$= \frac{\eta}{\ell t^2} (\overline{\tau^2} + \overline{\lambda^2} - 2\ell^2) .$$

Since $\overline{(v_1 - v_2)^2} = 2\overline{v^2} - 2\overline{v_1 v_2}$, it follows that:

$$\frac{\partial(\overline{v_1 v_2})}{\partial \eta} = - \frac{\overline{\tau^2} + \overline{\lambda^2} - 2\ell^2}{2\ell t^2} .$$

The result obtained above is in accordance with this formula, since we find $\overline{\lambda^2} = 2\ell^2$ with the expression taken for f_1.

It follows that we must require that the arrangement of the τ_i and λ_i shall satisfy the condition:

$$\text{(38a)} \qquad \frac{1}{N} \overline{\left[\sum_{k=1}^N (\tau_{i+k} - \lambda_{i+k}) \right]^2} \longrightarrow 0 .$$

It can be expected that this condition will be connected in some way with (38). A general analysis of the connection has not been made, but a particular way to satisfy (38) is to require that the τ_i and λ_i separately satisfy the conditions:

$$\frac{1}{N}\overline{\left[\sum \tau_{i+k} - N\ell\right]^2} \rightarrow 0; \qquad \frac{1}{N}\overline{\left[\sum \lambda_{i+k} - N\ell\right]^2} \rightarrow 0.$$

The first one leads to:

$$\omega + 2\sum \omega_k = 0.$$

The second one leads to a similar relation for the λ_i. However, when an additional supposition is made it can also lead to:

$$\sum g_k = \frac{\eta}{\ell} - \frac{1}{2}.$$

In that way (38) would be fulfilled.

A further investigation is desirable. There may be also a connection with the problem whether J will be zero under certain circumstances.

66. The complete expressions for $\overline{v_1 v_2}$ and $\partial(\overline{v_1^2 v_2})/\partial\eta$, valid for values of η large compared with $\nu t/\ell$, are:

$$(39) \qquad \overline{v_1 v_2} = \overline{v^2} - \frac{\ell^2}{t^2}\left\{\frac{1+\omega}{2}\frac{\eta}{\ell} - \frac{\eta^2}{2\ell^2} + \sum\left(\frac{\eta}{\ell}\varphi_k - X_k\right)\right\}$$

$$\frac{\partial}{\partial\eta}\left(\overline{v_1^2 v_2}\right) = -\frac{\ell^2}{t^3}\left\{\frac{1+\omega^*}{6} - (1+\omega)\frac{\eta}{\ell} + \frac{\eta^2}{\ell^2} - \sum\left(\frac{2\eta}{\ell}\varphi_k - X_k - \Phi_k\right)\right\}$$

(40)

where the functions φ_k, X_k, Φ_k are defined by:

$$\varphi_k(\eta) = \ell^{-2}\int_0^\eta d\lambda \; f_k(\lambda)\,\overline{\tau_i \tau_{i+k}}^{\,*}$$

$$X_k(\eta) = \ell^{-3}\int_0^\eta d\lambda \; f_k(\lambda)\,\lambda\,\overline{\tau_i \tau_{i+k}}^{\,*}$$

$$\Phi_k(\eta) = \ell^{-3}\int_0^\eta d\lambda \; f_k(\lambda)\,\overline{\left(\frac{1}{2}\tau_i + \tau_{i+1} + \ldots + \tau_{i+k-1} + \frac{1}{2}\tau_{i+k}\right)\tau_i \tau_{i+k}}^{\,*}.$$

The asterisk at the mean value signs indicates that, if necessary, the mean values should be considered as functions of the distance $\lambda =$
$= \xi_{i+k} - \xi_i = \lambda_{i+1} + \lambda_{i+2} + \cdots \lambda_{i+k}$.

It should always be kept in mind that all mean values introduced in these formulas are functions of the time.

Since the lower limit set to the value of η is very small, it is possible to define a domain in which the formulas can be developed into series according to powers of η. The first few terms of these series can be written:

$$(41) \qquad \overline{v_1 v_2} = \overline{v^2} - \frac{\eta}{\ell} \frac{\overline{\tau^2}}{2t^2} + \frac{\eta^2}{2\ell^2} \left(\frac{1}{t^2} - \frac{f_o}{\ell t^2} \overline{\tau_i \tau_{i+1}}^{*} \right) - \cdots$$

$$(42) \qquad \frac{\partial}{\partial \eta} \left(\overline{v_1{}^2 v_2} \right) = - \frac{\overline{\tau^3}}{6\ell t^3} + \frac{\eta}{\ell} \left\{ \frac{\overline{\tau^2}}{t^3} - \frac{f_o}{2\ell t^3} \overline{(\tau_i + \tau_{i+1}) \tau_i \tau_{i+1}}^{*} \right\} - \cdots$$

where f_o is the limiting value of $f_1(\lambda_i)$ for $\lambda_i \rightarrow 0$, while the asterisk now means that, properly speaking, the mean values should be calculated for $\lambda_i = 0$. (The fact that these formulas are not even functions of η is connected with the circumstance that they are not valid for very small values of η; they cannot be continued analytically to 0 or through 0.)

The expressions can be substituted into (31), from which the term multiplied by the viscosity ν must be omitted. Comparison of the terms independent of η on both sides brings us back to the ordinary dissipation equation; this can be considered as a first check on our results.

Comparison of the terms of the first degree in η leads to:

$$(43) \qquad \frac{d}{dt} \left(\frac{\overline{\tau^2}}{\ell t^2} \right) = - \frac{2\overline{\tau^2}}{\ell t^3} + \frac{f_o}{\ell t^3}$$

This equation can also be obtained by means of a direct calculation. It will be evident that the first term on the right-hand side is immediately connected with the presence of t^2 in the denominator on the left-hand side. To obtain the second term it is convenient to multiply by the constant length $S = N \cdot \ell$; we then have:

$$S\,\overline{\tau^2}/\ell = N\,\overline{\tau^2} = \sum{}' \tau_i^2 .$$

Now $\sum{}'\tau_i^2$ increases whenever two consecutive fronts merge, with an amount $2\tau_i\,\tau_{i+1}$. The frequency of this occurrence is given by the number of segments λ_{i+1} which become zero in unit time; this is given by $-N f_o(d\lambda_{i+1}/dt)$, where $d\lambda_{i+1}/dt$, for λ_{i+1} going to zero, has the value $-(\tau_i + \tau_{i+1})/2t$ (compare the expression for $d\lambda_i/dt$ given in Section 58, p. 131). Hence we obtain:

$$\frac{d}{dt}\left(\sum{}'\tau_i^2\right) = +N f_o \cdot \frac{\overline{(\tau_i + \tau_{i+1})\,\tau_i\,\tau_{i+1}}^{\,*}}{t} ,$$

which, after division by $S = N \cdot \ell$ explains the second term of the equation. It is gratifying that this comes out so well, for it provides another check on the calculations leading to the expressions for $\overline{v_1 v_2}$ and $v_1^2 v_2$ and on the rules for the merging together of vertical fronts.

67. An Objection Against the Applicability of the Similarity Hypothesis to the Solutions Considered Here. The expression obtained for the frequency of merging of vertical fronts can be applied to obtain some further results. The decrease of the average number of segments contained in a large length S and the corresponding increase of the average length ℓ of the segments are given by:

$$(44) \quad -\frac{1}{N}\frac{dN}{dt} = \frac{1}{\ell}\frac{d\ell}{dt} = f_o \, \frac{\overline{\tau_i + \tau_{i+1}}^{\,*}}{2t} .$$

It is also possible to check the expression for $d\overline{v^2}/dt$; to check the invariance of J; and to find the value of $d\epsilon/dt$.

More important is the following formula, deduced from (43) and (44):

$$\frac{d}{dt}\left(\frac{\overline{\tau^2}}{\ell^2}\right) = \frac{f_o}{\ell^2 t}\left\{ \overline{(\tau_i + \tau_{i+1})\,\tau_i\,\tau_{i+1}}^{\,*} - \frac{1}{2}\,\overline{(\tau_i + \tau_{i+1})}^{\,*}\;\overline{\tau^2} \right\} .$$

The quantity between brackets on the left-hand side is dimensionless; its value has been denoted by $1 + \omega$. It should be a constant if the arrangement would retain statistical similarity. However, it is difficult to see how the right-hand member of the equation can become zero unless the τ_i have to satisfy rather improbable conditions. If we suppose that the mean values to be used here are not different from ordinary mean values without the asterisk (i.e., for λ_i different from zero) and if further we suppose that consecutive τ_i are approximately independent, we find:

$$\frac{d}{dt}(1+\omega) \cong \frac{f_o \ell}{t}(1+\omega) .$$

This would point to a continuous increase with time of $1+\omega$, which is completely different from any approach to an arrangement which will remain statistically similar to itself.

The result is due to the circumstance that the equation for $d\lambda_i/dt$, obtained in Section 58, gives a big chance for shrinking to zero to those segments λ_i, for which either τ_i or τ_{i-1} is large. On the whole, therefore, wherever we find large τ_i, there is a good chance that these will quickly combine with other τ's to form larger and larger τ's. Since small τ_i will not combine at the same rate, there is tendency towards a greater spreading of values. Of course, such a qualitative argument is not sufficient to settle the problem, since there is a complicated interplay between the τ_i and the λ_i, and it is also possible that the value of f_o may decrease. Attempts have been made to formulate the problem in terms of a distribution function and a Boltzmann equation, but these attempts so far have not been successful.

It may be of interest to mention that the laws of motion of the vertical fronts of these solutions can be illustrated with the aid of a molecular analogue in which it is assumed that molecules move along a line of infinite extent and that they combine whenever there is a collision. We denote the coordinates of the molecules by ξ_i; the velocities by $d\xi_i/dt = d\zeta_i/dt = \zeta_i/t$; the masses by τ_i; and the momenta by $\tau_i \zeta_i/t$. At every collision masses and momenta are added; kinetic energy is lost. The laws of motion for the molecules then conform to those of the fronts. The number of molecules in every finite domain of the line on which they are moving decreases continually; however, when the domain is infinite, statistical considerations still can be applied, and it would be possible to ask whether there might be an approach to a definite pattern of mass distribution.

For such a molecular system our result would be that no definite pattern would be approached, but that on the contrary also here there would be found a tendency towards increased spreading.

68. Correction of Formula (39) to Make it Valid for Very Small Values of η. In the deduction of form. (37) the fronts had been treated as if the vertical slope did not occupy a finite length of the y-axis, but was of infinite steepness. The same supposition was used in the deduction of formulas (39) and (40), to which reference was made. More accurate expressions can be obtained when use is made of formula (8) for the course of v at a steep front, or of its equivalent, formula (21). The integral of $v_1 v_2$ (in which η is supposed to be of the order $\nu t/\tau_i$) is calculated with the aid of this expression over an interval of the y-axis extending to both sides of the front at ξ_i, and having a length of normal order of magnitude but not including any neighboring front. The difference of the result obtained with a finite (though small) value of ν and the limit for $\nu = 0$ gives the correction connected with the particular front considered. This correction appears to be:

$$\frac{\eta\,\tau_i^2}{2t^2}\left(1 - \operatorname{ctnh}\frac{\tau_i \eta}{4\nu t}\right).$$

The correction to be applied to (39) is the mean value of this expression:

$$\frac{\eta}{2\ell t^2}\left(\overline{\tau_i^2} - \overline{\tau_i^2 \operatorname{ctnh}\frac{\tau_i \eta}{4\nu t}}\right).$$

For values of η small in comparison with $4\nu t/\ell$ we may even develop the hyperbolic cotangent function, which gives:

$$\frac{\eta}{2\ell t^2}\left(\overline{\tau_i^2} - \frac{4\nu t}{\eta}\,\overline{\tau_i} - \frac{\eta}{12\nu t}\,\overline{\tau_i^3}\ \ldots\right) =$$

$$= -\frac{2\nu}{t} + \frac{\eta \ell}{2t^2}(1+\omega) - \frac{\eta^2 \ell^2}{24\nu t^3}(1+\omega^*)\ \ldots\ .$$

In this way it is seen that there is also a small correction to the value of $\overline{v^2}$ given in (25). The expression obtained for $\overline{v_1 v_2}$, in the domain $|\eta| \ll 4\nu t/\ell$, is:

$$\overline{v_1 v_2} = (\overline{v^2})_{\text{uncorrected}} - \frac{2\nu}{t} - \frac{\eta^2 \ell^2}{24\nu t^3}(1+\omega^*)\ \ldots\ . \tag{45}$$

The term with η^2 in (39) and the terms depending on the functions φ_k and X_k have been omitted, since, for the values of η considered here, they are insignificant in comparison with the term appearing in (45). It will be seen that the linear term has disappeared.

We can now apply Eq. (34) and obtain:

$$\epsilon = \nu \left(\overline{\frac{\partial v}{\partial y}}\right)^2 = \frac{\ell^2}{12\, t^3} (1 + \omega^*)$$

in conformity with (26).

69. Kolmogoroff's Similarity Theory for the Correlation Function.

Kolmogoroff has enunciated the hypothesis that for distances η small compared with a length, which is equivalent to our ℓ, the correlation function should be determined only by the magnitude of the energy dissipation ϵ and the kinematic viscosity ν. The quantity ℓ, which has played a prominent part in the formulas referring to our solutions, for such values of η should be unimportant.

Kolmogoroff defines the correlation by means of the quantities:

$$\overline{(v_1 - v_2)^2} \quad ; \quad \overline{(v_1 - v_2)^3}$$

where again v_1, v_2 represent the values of v in two points, y and $y+\eta$, respectively. The mean value is taken with respect to y for a fixed value of η. These quantities are connected with those used in the preceding sections by means of the equations:

$$\overline{(v_1 - v_2)^2} = 2\,(\overline{v^2} - \overline{v_1 v_2}) \;;$$

$$\overline{(v_1 - v_2)^3} = 6\; \overline{v_1^2 v_2} \;.$$

Application of dimensional reasoning on the hypothesis that no other physical quantities are relevant than ϵ and ν, leads to the formulas:

$$\overline{(v_1 - v_2)^2} = \nu^{1/2}\, \epsilon^{1/2}\, F_1(\eta \epsilon^{1/4}\; \nu^{-3/4})$$

$$\overline{(v_1 - v_2)^3} = \nu^{3/4} \epsilon^{3/4} F_2(\eta\epsilon^{1/4} \nu^{-3/4}) ,$$

where F_1 and F_2 are unknown functions.

Kolmogoroff further supposes that if the Reynolds number for the field $Re = \ell^2/\nu t$ is sufficiently high, there should be a domain in which $\eta\epsilon^{1/4} \nu^{-3/4}$ is a large number and in which these expressions would take forms <u>independent</u> of the viscosity. This prescribes a certain condition for the functions F_1 and F_2 and leads to the results:

$$\overline{(v_1 - v_2)^2} = C_1 (\epsilon \eta)^{2/3}$$

$$\overline{(v_1 - v_2)^3} = C_2 \epsilon\eta ,$$

C_1 and C_2 being constants.

From formula (40), taken in connection with the expression for ϵ given in (26), it will be seen that the second result is quite acceptable; the value of C_2 appears to be -12.

The first result, however, is so much different from that obtained in (39), that it is difficult to judge its validity. It may be a useful approximation over a certain domain, but sufficient evidence for this cannot be obtained. Nevertheless it is of importance to observe that Kolmogoroff's expression for $\overline{(v_1 - v_2)^2}$ definitely increases less rapidly with η than the result which is obtained on the assumption that the τ_i and λ_i should be completely independent of each other. (Compare the considerations developed in Section 65.) Hence in order that Kolmogoroff's formula may be valid, it is necessary that there exist certain relations between consecutive τ_i and λ_i, of such nature that the condition (38a) shall be satisfied. This seems to be an important point, which is not restricted to the solutions considered here, but has its bearing on the hydrodynamical problem.

It is perhaps possible to give the following turn to Kolmogoroff's reasoning: Consider a great number of cases of our type of solution, starting from various initial stages, so that there will be a great variety of values of ℓ and of "ages" t.

Now take together all those cases for which ϵ has the same value. This will require a certain relation between the value of ℓ and the age. If we should return for a moment to the supposition of statistical similarity, proportionality between ℓ^2 and t^3 would be required, so that "age" $\sim \ell^{2/3}$. We then obtain: $\overline{v^2} \sim \ell^2/t^2 \sim \ell^{2/3}$.

If we now assume that for values of η, i.e., small compared with ℓ and large compared with $\nu t/\ell$, the curve for $\overline{(v_1 - v_2)^2}$ should be the same for all cases having the same ϵ, then it would be necessary that in the range specified we should have $\overline{(v_1 - v_2)^2} \sim \eta^{2/3}$. The reasoning looks rather artificial.

It is to be observed that with Kolmogoroff's hypotheses the value of $\overline{v^2}$ is not independent of ν, but is given by: constant $\cdot \nu^{1/2} \epsilon^{1/2}$.

Energy Transfer Through the Spectrum

70. Spectral Analysis of the Solutions. The general idea of the application of Fourier analysis to functions of a single variable y and the connection with the correlation function has been considered in Chapter III, Sections 16 and 17. The results given there can be applied to the present case. According to formula (4) of that section, we write:

$$\overline{v_1 v_2} = \int_0^\infty \Gamma(k) \cos k\eta \, dk \tag{46}$$

For $\eta = 0$ this gives the energy spectrum:

$$E = \frac{1}{2}\overline{v^2} = \frac{1}{2}\int_0^\infty \Gamma(k)\, dk$$

Making use of formula (6) of Section 17, we have:

$$\Gamma(k) = \frac{2}{\pi}\int_0^\infty \overline{v_1 v_2} \cos k\eta \, dk . \tag{47}$$

Hence we see that

$$\Gamma(0) = \frac{2}{\pi} J$$

and if we develop the cosine-function we obtain:

(47a) $$\Gamma(k) = \frac{2}{\pi} J - \frac{k^2}{2} \Gamma_2 + \dots .$$

where

$$\Gamma_2 = \frac{2}{\pi} \int_0^{\infty} \overline{v_1 v_2} \cdot \eta^2 \, d\eta .$$

It does not seem easy to obtain expressions for the function $\Gamma(k)$ or quantities like Γ_2 connecting them with the statistical characteristics of the "saw tooth" profile for v. However, when k is large compared with $1/\ell$ and still small in comparison with $\ell/\nu t$, we can obtain the approximation:

(48) $$\Gamma(k) \cong \frac{\overline{\tau^2}}{\pi k^2 t^2 \ell} ,$$

where the error is of the order k^{-3} or k^{-4}. One would be tempted to combine (47a) and (48) into an expression of the type:

$$\Gamma \cong \frac{A}{1 + B k^2 \ell^2} ,$$

but the data available do not permit proof of this expression.

If k becomes of the order $\ell/\nu t$ it appears that (48) must be replaced by:

(48a) $$\Gamma(k) \cong \frac{4\pi \nu^2}{\ell} \overline{\left(\sinh \frac{2\pi k \nu t}{\tau_i} \right)^{-2}} ,$$

although further mathematical investigations are necessary. From (48a) it would follow that at the end of the spectrum $\Gamma(k)$ decreases exponentially with k. Existing theories concerning the course of $\Gamma(k)$ mostly give a decrease according to some inverse power law. On general mathematical grounds it is likely, however, that ultimately an exponential decrease must be obtained, since otherwise the course of v, and in the hydrodynamical case, the course of the flow, in dimensions of the order $\nu t/\ell$ could not be smooth. This point is stressed by J. von Neumann in a report on "Recent Theories of Turbulence".

From Kolmogoroff's similarity hypotheses it has been deduced that in a certain domain of k-values one should have:

$$\Gamma(k) \sim \epsilon^{2/3}\ k^{-5/3}\ .$$

This, of course, differs notably from (48). The two results can, perhaps, be brought into a relation if we use the artifice mentioned in the preceding section. We consider a set of cases for which ϵ has the same value. As observed, for the fields selected in this way we must then have "age" (t) $\ell^{2/3}$. From (48) we deduce:

$$\Gamma \sim \frac{\ell}{k^2 t^2} \sim \frac{1}{k^2\, \ell^{1/3}}\ .$$

If we assume that, within the domain of k-values mentioned above, (I) $\Gamma(k)$ is a function of $k\ell$, and (II) that for all fields with the same ϵ the curves for $\Gamma(k)$ should coincide, it would be necessary that Γ should be proportional to $k^{-5/3}$. The reasoning, however, is just as artificial as the one given before.

A result obtained by Heisenberg for hydrodynamical turbulence, viz., that Γ must become proportional to k^4 when k decreases to zero, has as its analogue in our solutions that Γ must approach to a constant value for $k \rightarrow 0$. This is in accordance with what has been found above.

71. Energy Transfer Through the Spectrum. We introduce the Fourier transform of the correlation function $\overline{v_1^2 v_2}$:

$$\overline{v_1^{\,2} v_2} = \int_0^{\infty} \psi(k) \sin k\eta\ dk$$

with:

$$\psi(k) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \overline{v_1^{\,2} v_2}\ \sin k\eta\ d\eta\ .$$

It is then possible to transform von Karman's and Howarth's equation (31) into:

$$(49) \qquad \frac{\partial \Gamma}{\partial t} = k\,\psi - 2\nu k^2\, \Gamma$$

For $k = 0$ this gives $\partial\Gamma(0)/\partial t = 0$, which is nothing else than the constancy of Loitsiansky's invariant.

The equation for the energy transfer through the spectrum is obtained from (49) by integration with respect to k:

$$\frac{d}{dt}\int_0^k \Gamma\, dk = \int_0^k k\,\psi\, dk - 2\nu \int_0^k k^2\, \Gamma dk. \tag{50}$$

The term on the left-hand side gives the change with time of (twice the) energy in the part of the spectrum which extends from 0 to k. The last term on the right-hand side determines the amount that is lost by viscous dissipation. The other integral, when taken with the sign reversed, gives the energy that is transmitted from the part 0 .. k to the part beyond k. Since we have

$$\frac{\partial}{\partial\eta}\overline{v_1^{\,2}v_2} = \int_0^\infty k\psi \;\cos k\eta\; dk,$$

and since the derivative of $\overline{v_1^{\,2}v_2}$ with respect to η is zero for $\eta = 0$, it follows that:

$$\int_0^\infty k\psi\; dk = 0.$$

Hence the term with ψ does not bring over-all gain or loss of energy; it represents pure interchange.

To find ψ we follow a similar method as was used in Section 17 and calculate

$$\overline{v_1^2 v_2} = \frac{1}{2M}\int_{-M_1}^{+M_1} dy \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_3\, \varphi(k_1)\,\varphi(k_2)\,\varphi(k_3)\, e^{i(k_1+k_2+k_3)y+ik_3\eta}$$

$$= \frac{1}{M}\int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_3\,\varphi(k_1)\,\varphi(k_2)\,\varphi(k_3)\, e^{ik_3\eta}\;\frac{\sin(k_1+k_2+k_3)\, M_1}{k_1+k_2+k_3}$$

When M_1 is made infinite this gives:

$$\overline{v_1^{\,2} v_2} = \frac{\pi}{M}\int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2\; \varphi(k_1)\;\; \varphi(k_2)\;\; \varphi(-k_1-k_2)\; e^{-i(k_1+k_2)\eta}.$$

In the double integral we shall write: $k_1+k_2 = k$; $k_1 = k'$. It then transforms into:

$$\overline{v_1^2 v_2} = \frac{\pi}{M} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dk' \;\varphi(k')\,\varphi(k-k')\,\varphi(-k)\, e^{-ik\eta} .$$

On the other hand, we bring (49) in the form:

$$\overline{v_1^2 v_2} = -\frac{1}{2i} \int_{-\infty}^{+\infty} \psi(k)\, e^{-ik\eta}\, dk ,$$

assuming $\psi(-k) = -\psi(k)$. Comparison now leads to the following expression for ψ :

$$\psi(k) = -\frac{2\pi i}{M} \int_{-\infty}^{+\infty} dk' \;\varphi(k')\,\varphi(k-k')\,\varphi(-k) .$$

72. It will be seen that the value of ψ depends on an integral of a product of three factors φ the arguments of which have an algebraic sum equal to zero. Every frequency k comes into connection with all other frequencies, both smaller and larger. From the expression for ψ alone it is not to be seen whether energy transport will be mainly from small k to large k, or inversely, or both ways simultaneously.

It was observed in Section 17 that the absolute value of φ is proportional to $M^{1/2}$. In order that ψ shall be independent of M (as it should be) it is therefore necessary that the outcome of the integration brings a factor $M^{-1/2}$, since there is only the factor M^{-1} before the integral. Looking at the dimensions, each function φ has the dimensions (velocity · length), while the function ψ must have the dimensions $(\text{velocity})^3$ · (length). Hence an extra factor $M^{-1/2}$ must be accompanied by a factor of dimension $(\text{length})^{1/2}$. The most probable supposition is that the full factor should be $(k\cdot M)^{-1/2}$.

How could the appearance of such a factor be explained? The explanation can be sought in the effect of phase relations. The circumstance that the algebraic sum of the arguments of the three functions φ is equal to zero, might in itself not guarantee a stationary phase for the product. The phase factor of the product could rapidly change with k', so that the contribution to the integral for most values of k' would be negligible. A stationary phase, however, might appear with certain simple rational relations between the three frequencies involved, for instance:

$$k_1 = k - k_1 = k/2;$$

$$k_1 = k/3 \text{ and } k_1 = 2k/3, \text{ etc.}$$

$$k_1 = 2k; \; k - k_1 = -k, \text{ etc.}$$

We may suppose that the "resonance breadth" on the k-scale would be proportional to $k \cdot (kM)^{-1/2}$ (with different numerical factors for the various cases). The result of the integration would then be of the form:

$$\psi = \frac{k^{1/2}}{M^{3/2}} \sum a_m \left| \varphi_i \right| \cdot \left| \varphi_2 \right| \cdot \left| \varphi_3 \right| ,$$

where the arguments of $\varphi_1, \varphi_2, \varphi_3 \equiv$ would be in some simple rational relation, while the a_m would be numerical constants. Expressing the absolute values of the φ by means of the Γ according to Eq. (3) of Section 17 and considering $k\psi$, which is the actual term occurring in Eq. (50), we would find:

$$k\psi = \left(\frac{k}{2\pi}\right)^{3/2} \sum a_m (\Gamma_1 \Gamma_2 \Gamma_3)^{1/2} .$$

The most important term might be:

$$\left(\frac{k}{2\pi}\right)^{3/2} a_1 \; \Gamma\left(\frac{k}{2}\right) \; \sqrt{\Gamma(k)} .$$

This reasoning is of highly speculative, but the supposition that transfer of energy would mainly take place when the three frequencies involved make some "harmonic chord", would have the consequence that the number of steps involved in transporting energy over a certain region of the spectrum would be roughly proportional to the logarithm of the ratio of the frequencies limiting this region. Such a supposition has been made by L. Onsager in a note on turbulence, where he speaks of the transfer of energy as a "cascade process."

Solutions Representing Turbulence Coupled with a Primary Motion

73. The solutions of Eq. (1) considered thus far were subjected to decay. In order to be able to form solutions deriving energy from an outside source, in such a way that a stationary state of turbulence is maintained of a type analogous to that obtained in pipe flow, we replace Eq. (1) by an extended equation:

$$(51) \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2} + \frac{Uv}{b} .$$

In the term added on the right-hand side U stands for a given quantity, an analogue of the average velocity of flow over the cross section in the case of pipe flow. The term has been given the form of a product of U and v, with a factor b in the denominator to give it the proper physical dimensions. In this way Eq. (51) can be satisfied by taking $v = 0$ and turbulence can derive energy from the main motion only when it is already in existence. The same is true of hydrodynamic turbulence, although in that case the coupling with the main motion depends on terms containing derivatives. (Compare Eq. (9) of Chapter VI, p. 72.)

We limit the domain of y to $0 \leq y \leq b$ and subject v to the condition:

$$v = 0 \text{ at } y = 0 \text{ and } y = b.$$

It is possible to consider U itself as a variable quantity, depending on the time. We then subject U to the equation:

$$(52) \qquad \frac{dU}{dt} = \frac{P}{b} - \frac{\nu U}{b^2} - \frac{1}{b^2}\int_0^b v^2\,dy$$

Now the parameter P is assumed to be a given quantity. It can be taken as the analogue of the pressure in pipe flow. The equation expresses that a motion with the velocity U experiences both "laminar" friction, given by the term $\nu U/b^2$, and a resistance derived from the turbulence, given by the integral. Since we had assumed that U does not depend on y, it has been inevitable to choose a rather artificial form for the resistance terms. Nevertheless they serve their purpose and they allow the formation of an equation of energy for the "main motion" U and the "turbulence" v together, as follows:

$$(53)\qquad \frac{d}{dt}\left[\frac{1}{2}\,b\,U^2 + \frac{1}{2}\int_0^b v^2\,dy\right] = PU - \frac{\nu U^2}{b} - \nu\int_0^b\left(\frac{\partial v}{\partial y}\right)^2 dy.$$

The quantity between the brackets on the left-hand side represents the kinetic energy in the field. On the right-hand side we have the energy derived from the pressure acting on the main motion, and losses due to the laminar friction of the main motion and to the dissipation in the turbulence. The terms of the second degree in Eqs. (51) and (52) have disappeared; they represented a coupling without loss or gain.

74. Stationary Solutions of the System (51) - (52). The system admits two types of stationary solutions. In one type $v = 0$; we shall say that they represent "laminar" flow without turbulence. In the other type v is different from zero, and although it is stretching the meaning of the word "turbulence", we shall use it for any solution in which v is not permanently and everywhere zero.

The "laminar" solution is given by:

$$(54)\qquad U = Pb/\nu; \quad v = 0.$$

It is seen that U is proportional to P in this solution.

The stability of the solution can be investigated by means of the method of small disturbances. The necessary formulas will come automatically in Section 78, below, where it is found that the laminar solution is stable so long as $U < \pi^2 \nu/b$. When U exceeds this value, the solution becomes unstable and gives way to one in which v is different from zero. It can be observed that the dimensionless parameter $U\,b/\nu$ plays the part of a Reynolds number for the system (51) - (52), with the critical value π^2.

To obtain stationary "turbulent" solutions, we start from (51) without the time derivative:

$$(55)\qquad \nu\frac{d^2v}{dy^2} - v\frac{dv}{dy} + \frac{Uv}{b} = 0,$$

where U is a constant. We introduce an auxiliary variable

$$\eta = -\frac{b}{U}\frac{dv}{dy}$$

(the η used here has nothing to do with the η occurring in correlation functions). The following equation is obtained for η:

$$\frac{\nu U}{b}\,\eta\,\frac{d\eta}{dv} + v(1+\eta) = 0 .$$

This equation can be solved. The complete solution of (55) can be expressed in parametric form, as follows:

$$v = \sqrt{\frac{2\nu U}{b}}\,\sqrt{C - \eta + \ln(1+\eta)}$$

$$y = \sqrt{\frac{\nu b}{2U}}\int \frac{d\eta}{1+\eta}\qquad\frac{1}{\sqrt{C - \eta + \ln(1+\eta)}} ;$$

with C as the integration constant. The variable η must be situated between two limits η_1, η_2, given by the roots of

$$C - \eta + \ln(1+\eta) = 0 .$$

When C is large, these roots are approximately equal to:

$$\eta_1 \cong -1 + e^{-C-1} ; \qquad \eta_2 \cong C + \ln(C+1).$$

We take η_1 as the lower limit of the integral; since $\eta = \eta_1$ gives $v = 0$, this can be made to correspond to $y = 0$. As v must be zero also at $y = b$, a condition must be satisfied which fixes the value of C. Since there may be intermediate zeros the condition takes the form:

$$\sqrt{\frac{\nu b}{2U}}\int_{\eta_1}^{\eta_2} \frac{d\eta}{1+\eta}\qquad\frac{1}{\sqrt{C - \eta + \ln(1+\eta)}} = \frac{b}{m} ,$$

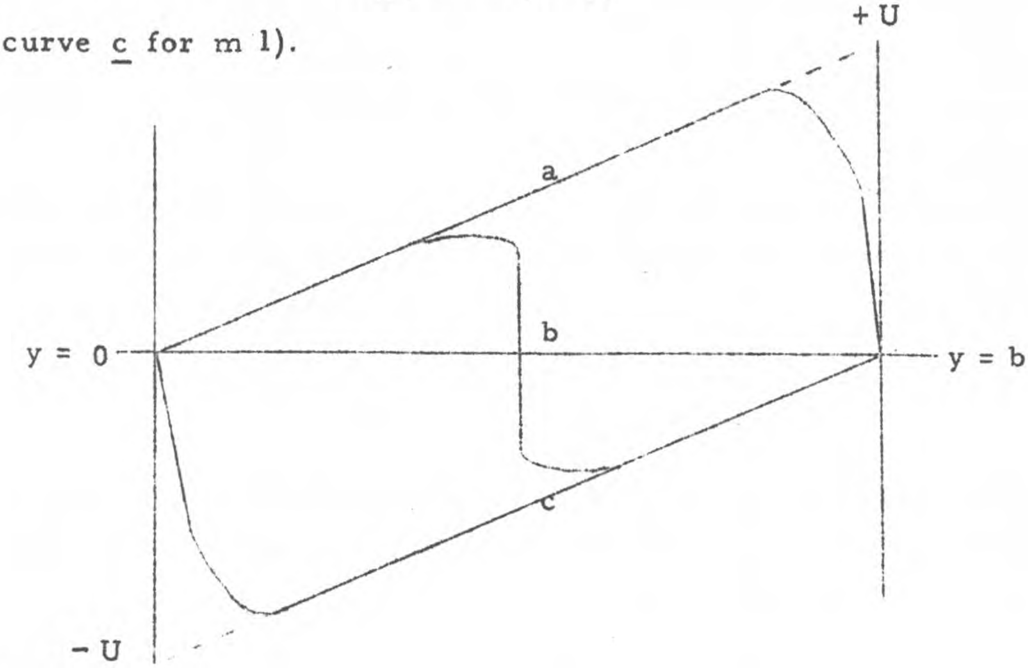
in which m must be a positive integer. The simplest solution is obtained with $m = 1$. See the accompanying diagram, curve a, for the case of a large value of C. A solution with $m = 2$ is represented by curve b.

The value of the definite integral decreases when C decreases. Its minimum value, obtained for $C \to 0$, is $\pi\sqrt{2}$. Hence there is a maximum for m, determined by the largest integer contained in $(Ub/\pi^2\nu)^{1/2} = (R_e/\pi^2)^{1/2}$.

There are other series of solutions, for instance the series obtained from the one just considered by means of the formula:

$$v(y) = -v(b-y)$$

(See curve $\underline{c}$ for m 1).



75. Approximate Formulas for the Stationary Solutions. We assume Ub/ν to be large. In this case the solutions with small m (for which C is large) can be represented by means of approximate expressions. A non-rigorous but convenient method to obtain these approximations is to divide the domain $0 \leq y \leq b$ into regions where the term $\nu(d^2v/dy^2)$ can be neglected and where we find:

$$v \cong \text{constant} + Uy/b;$$

and into regions where the term Uv/b is relatively unimportant in comparison with the other two terms and where we find:

$$v \cong -A \tanh A(y-B)/2\nu ,$$

A and B being constants. The constants must be adjusted in such a way that the boundary conditions at $y = 0$ and $y = b$ are satisfied and that no discontinuities are obtained in the course of y. We mention the following examples, (the expressions combine the two partial solutions):*

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*The deduction of the approximations had been given in a paper "Mathematical Examples Illustrating Relations Occurring in the Theory of Turbulent Fluid Motion," Verhandl Kon. Nederl. Akad. Wetenschappen Amsterdam, Afd. Natuurk. (1st Sect.) Vol. 17, No. 2, pp. 1 - 53 (1939). (Footnote continued on next page.)

$$(m = 1): \quad v = U\left(\frac{y}{b} - \tanh\frac{Uy}{2\nu}\right) \qquad (\text{curve } \underline{c})$$

$$(56) \quad (m = 1): \quad v = U\left\{\frac{y}{b} - 1 + \tanh\frac{U(b-y)}{2\nu}\right\} \qquad (\text{curve } \underline{a})$$

$$(m = 2): \quad v = \frac{U}{2}\left\{\frac{2y}{b} - 1 - \tanh\frac{Uy}{4\nu} + \tanh\frac{U(b-y)}{4\nu}\right\} \qquad (\text{curve } \underline{b})$$

The regions where the course of v is determined by the tanh-function present a steep gradient and are the only regions which materially contribute to the dissipation. In the case of the first one of the solutions given above we find:

$$\nu \int \left(\frac{\partial v}{\partial y}\right)^2 dy = U^3/3 \; .$$

When this result is substituted into the energy equation (53), in which the left-hand member is zero for a stationary solution, while in the right-hand member we neglect $\nu U^2/b$, we obtain:

$$(57) \qquad P = U^2/3 \; .$$

The same expression can be deduced from Eq. (52), in which likewise the left-hand member is zero, while the term $\nu U/b$ can be neglected. We thus arrive at the result that <u>with stationary turbulence the relation between U and the pressure leads to a quadratic resistance law</u>.

76. <u>Nonstationary Solutions</u>. If U is considered as a constant, nonstationary solutions of Eq. (51) can be obtained without great difficulty. In order to obtain an approximate picture, we can follow a similar way as in the preceding section. We divide the domain for y into regions where we

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(Continued from previous page)

In that paper the parameter b had been replaced by unity, while on the other hand a factor 2 had been inserted before the term $v(\partial v/\partial y)$, which causes certain numerical differences in the expressions. Part of this paper has been taken over, with some new material, in "A Mathematical Model Illustrating the Theory of Turbulence," Advances in Appl. Mechanics, Vol. 1, pp. 171-199, (1948). In the latter paper (which does not reproduce the deduction of the approximations), the parameter b had been used and the factor 2 before $v(\partial v/\partial y)$ was retained. The approximations are straightforward and can easily be reconstructed.

can neglect the viscosity and simplify Eq. (51) to:

$$(58) \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \frac{Uv}{b} ;$$

and other regions where the approximation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2}$$

will be appropriate. The latter equation is identical with Eq. (1) and the method to be followed is the one indicated in Section 52 (pp. 116-118), with the result given already in Eq. (8) of that section.

Equation (58) can be solved by means of its characteristics. We can, however, also start in a way similar to that followed in Section 50 and look for a solution of the form. (2): $v = \beta\,(y - \sigma)$. We then obtain:

$$\text{either} \quad v = \frac{U}{b}(y - \sigma)\,\frac{\epsilon}{1+\epsilon} ; \quad \text{or} \quad v = -\frac{U}{b}(y - \sigma)\,\frac{\epsilon}{1-\epsilon} ,$$

where $\epsilon = e^{U(t - t_o)/b}$, t_o being an integration constant. The first form is appropriate when the initial slope β is positive; it is seen that for $t \to \infty$ the slope ultimately approaches to U/b. The second form must be used when the initial slope is negative; the adjustment of t_o will then require that initially ϵ is smaller than unity. With increase of t there comes an instant when the slope threatens to become infinite. We must then correct the result by making use of the appropriate form of Eq. (8) mentioned before.

Attention must be given to the conditions at the limits $y = 0$ and $y = b$. We shall not go into details and mention only that as far as can be seen by means of the approximate method, solutions starting from arbitrary initial conditions approach asymptotically to one of the stationary solutions found before.

77. Application of Fourier Analysis. Having regard to the boundary conditions we can develop v into a Fourier series:

$$(59) \qquad v = -\sum \xi_n \sin\frac{\pi n y}{b}$$

where, in general, the coefficients or amplitudes ξ_n will be functions of the time. The minus sign before the sum is of no importance in itself, but has been introduced to obtain positive values for the amplitudes in the case of the solution c) of Section 75. The stationary solutions considered in that section have Fourier series with amplitudes which are independent of the time. The amplitudes can be calculated with sufficient accuracy from the approximate formulas given in that section. For the three cases c), a), b) the following results are obtained:

$$(60) \qquad \xi_n = \frac{2\pi\nu}{b \sinh(n\pi^2\nu/Ub)} \quad \text{(for c)}$$

$$\xi_n = \frac{(-1)^n\, 2\pi\nu}{b \sinh(n\pi^2\nu/Ub)} \quad \text{(for a)}$$

$$\left.\begin{aligned} \xi_n &= \frac{4\pi\nu}{b \sinh(2\pi^2 n\nu/Ub)} && \text{if } n \text{ is even} \\ &= 0 && \text{if } n \text{ is odd} \end{aligned}\right\} \quad \text{for b)}$$

It will be seen that these amplitudes ultimately decrease as exponential functions of the index n. Taking the first case, we have:

$$(60a) \qquad \xi_n \cong \frac{4\pi\nu}{b} e^{-n\pi^2\nu/Ub} \quad \text{for large } n.$$

In the case a) the amplitudes have alternating signs; in the case b) the amplitudes of odd order are zero, but the general principle of exponential decrease is retained. It will be remembered that an exponential decrease was also surmised in the case of the Fourier development of decaying homogeneous turbulence, considered in Section 70.

It is possible to translate the partial differential equation (51) into an infinite system of simultaneous ordinary differential equations for the amplitudes ξ_n. For this purpose we substitute (59) into (51) and apply a simple transformation to the terms obtained from $v(\partial v/\partial y)$. It is then possible to separate the sine functions of various order and the following result is obtained:

$$(61) \qquad \frac{d\xi_n}{dt} = \left(\frac{U}{b} - \frac{n^2\pi^2\nu}{b^2}\right)\xi_n + \frac{n\pi}{2b}\left(\frac{1}{2}\sum_{1}^{n-1}\xi_h\,\xi_{n-h} - \sum_{1}^{\infty}\xi_h\,\xi_{n+h}\right)$$

In principle the development of any turbulent solution from a given initial state is determined just as well by the system of equations (61) as it is by the partial differential Eq. (51). No direct method for attacking the system (61) has been found thus far, but some important properties of it are easily recognizable.

78. In the first place we note that the system (61) is satisfied when all amplitudes ξ_n are zero. This means that there is no "turbulence"; we have "laminar" flow and the auxiliary equation gives a linear relation between the value of U and the pressure gradient P.

Looking at the terms on the right-hand side of (61), we note that the first term is linear in ξ_n and contains U/b as factor. This term represents the coupling between the "mean motion" U and the turbulent amplitude ξ_n. If this term were the only one on the right-hand side, an exponential increase of ξ_n with time would result. The circumstance that all amplitudes ξ_n have the same factor U/b is accidental for the model. This does not represent a feature of general importance.

Next follows a term also linear in ξ_n, which represents the damping effect of viscosity. The factor n^2 occurring in this term exhibits a general feature which is always obtained with Fourier components. Since the sign of the term is negative, it causes an exponential damping which increases rapidly with increasing order.

If, on the supposition of small amplitudes, we provisionally neglect the nonlinear terms in (61), the system reduces to a set of separate equations for the various amplitudes, without coupling between them. We immediately see that so long as $U < \pi^2 \nu/b$ (Reynolds number $Ub/\nu < \pi^2$) all components ξ_n, if stimulated, will be damped. When U exceeds the critical value, one or more of these components, when stimulated, will increase exponentially while the rest would show exponential decrease. This explains the transition from stability to instability of the laminar solution, to which reference had been made in Section 74. We also see that as soon as the increasing components have obtained a certain amplitude, the nonlinear terms can no longer be neglected and coupling between all components sets in. This coupling makes it possible that energy is detracted from the components which otherwise increase exponentially, so that their increase is reduced or ceases, while components which otherwise would disappear through viscous damping can gain energy and may remain in existence.

Two features of (61) now are of great importance. One has been mentioned already, viz., that the viscous damping has the factor n^2. The other feature is the presence of the sum

$$\sum_{1}^{n-1} \xi_h \, \xi_{n-h}, \quad \text{which represents a "convolution" of}$$

the series of amplitudes ξ_n. If we ask for a term which can balance the viscous damping for very large n, that term must have the factor n^2. Now such a term can be obtained if we assume that the "convolution" $\Sigma \xi_h \xi_{n-h}$ becomes a linear function of n for large n. Since the convolution is already multiplied by a factor n, we then obtain the required term with n^2. It is easily seen that this result can be secured by supposing that for large n the amplitudes ξ_n depend exponentially on n. In that case the product $\xi_h \xi_{n-h}$ will become independent of h and the convolution will contain a number of such terms, which number increases linearly with n.

To make this reasoning more precise, we assume that:

$$\xi_n = \beta e^{-an} \quad \text{for} \quad n > N,$$

where N is some definite number. We suppose that there are no terms in the series of the ξ_n which drop out (as in the case of the solution b) - see Section 77). We then write out Eq. (61) for a value of n exceeding 2N (preferably greatly exceeding 2N). In the two sums occurring in (61) we take apart the terms which contain amplitudes ξ_h for which $h < N$ and amplitudes ξ_{n-h} for which $n - h < N$. We then obtain:

$$\text{(62)} \qquad \frac{d\xi_n}{dt} = \beta e^{-an} \left[\frac{U}{b} - \frac{n^2 \pi^2 \nu}{b^2} + \frac{n^2 \pi \beta}{4b} - \frac{\pi n}{2b} A \right]$$

where A is short for

$$\text{(62a)} \qquad A = \beta \left(N + \tfrac{1}{2}\right) - \sum_{1}^{N} \xi_h \left(e^{ha} - e^{-ha}\right) - \frac{\beta e^{-2(N+1)a}}{1 - e^{-2a}} .$$

It is not difficult to check that the expression for A does not change when the value of N is increased by one or more units.

The result obtained shows that the effect of viscosity can be balanced if we take:

$$\beta = \frac{4\pi\nu}{b} \tag{63}$$

which is in accordance with the formula mentioned before (60a).

This reasoning does not give us the value of α. If we look for a physical interpretation of β, the only feature which suggests itself is that:

$$\xi_h \xi_{n-h} = \beta\xi_n , \tag{64}$$

independently of n and h, provided both h and $n - h$ exceed N. This is a nonhomogeneous relation between amplitudes of a kind not found in linear problems.

79. If the condition (63) is satisfied, the term with n^2 disappears from the expression for $d\xi_n/dt$. There remains a term with n. It is possible to get rid of this term as well, if we make A vanish. This puts a condition on the amplitudes ξ_h for $h < N$. It will be seen that this condition can be satisfied to a large extent if we assume:

$$\xi_h = \frac{\beta}{e^{h\alpha} - e^{-h\alpha}} . \tag{65}$$

This formula passes into (63) when $h > N$, provided we can neglect $e^{-2N\alpha}$ in comparison with unity. It is seen that this new result brings us to a hyperbolic sine function of the type as occurred in (60).

To make A vanish rigorously, a more accurate calculation would be necessary. . But we can now make a start from the other side if we accept the expression (65) for small h and suppose that α is very small. In that case we have the approximation

$$\xi_h = \frac{\beta}{2h\alpha} . \tag{65a}$$

We can now substitute this approximation into (61), neglecting the viscous damping for small n. We have:

$$\sum_{1}^{n-1} \xi_h \, \xi_{n-h} = \frac{\beta^2}{2na^2} \left(1 + \frac{1}{2} + \dots \frac{1}{n-1} \right)$$

$$\sum_{1}^{\infty} \xi_h \, \xi_{n+h} = \frac{\beta^2}{4na^2} \left(1 + \frac{1}{2} + \dots \frac{1}{n-1} + \frac{1}{n} \right)$$

and (61) reduces to:

$$\frac{d\xi_n}{dt} = \frac{U}{b} \frac{\beta}{2na} - \frac{n\pi}{2b} \left(\frac{\beta}{2na} \right)^2 .$$

The two terms on the right-hand side compensate each other if:

$$\frac{\beta}{2a} = \frac{2U}{\pi} ,$$

giving:

$$\xi_n = \frac{2U}{n\pi} \quad \text{for small } n, \tag{66}$$

which again is in accordance with (60).

These considerations are far from a rigorous treatment of the system (61), and the last point in particular seems to be intimately connected with the peculiar structure of the model system.

Nevertheless it would appear that the reasoning leading to the asymptotic exponential law for the amplitudes ξ_n represents a general feature, not limited to the special form of the model system. It ensures the possibility of balancing viscous dissipation for very large n, that is, in the tail of the spectrum, by inflow of energy from a part of the spectrum preceding the component considered, in which the head of the spectrum (where the exponential formula does not apply) is of small importance. It does not appear necessary that a should be independent of the time; there might be, for instance, a slow increase with time. The actual value of a probably will depend on the character of the head of the spectrum, or, in other words, on the way in which energy is fed into the system.

80. The following additional remarks are of interest. If the Fourier series (57) is introduced into Eq. (52), we obtain:

$$(67) \qquad \frac{dU}{dt} = \frac{P}{b} - \frac{\nu U}{b^2} - \frac{1}{2b}\sum_1^{\infty}\xi_n^2$$

When U is independent of the time, the left-hand side is zero. It will be seen that the sum $\Sigma\xi_n^2$ can be calculated with sufficient accuracy from the approximation (66), giving $\Sigma\xi_n^2 = 2U^2/3$. Neglecting the term $\nu U/b^2$, we find:

$$P = U^2/3,$$

in conformity with a result already mentioned. We can interpret this result by saying that the reaction of the turbulence on the mean motion is dependent on the amplitudes of the components at the head of the spectrum only.

The energy equation (53) takes the form:

$$(68) \qquad \frac{d}{dt}\left[\frac{1}{2}bU^2 + \frac{1}{4}b\sum_1^{\infty}\xi_n^2\right] = PU - \frac{\nu U^2}{b} - \frac{\pi^2\nu}{2b}\sum_1^{\infty}n^2\xi_n^2 .$$

In this equation the approximation (66) is of no use, since it would make the sum $\Sigma n^2 \xi_n^2$ divergent. Indeed, the approximation (66) has the consequence that the dissipation has the same value for every component of the turbulence. The more accurate expression (60) is needed to make the sum convergent - in other words, to set a limit to the part of the spectrum where there is equipartition of dissipation.

For certain purposes it is useful to multiply Eq. (67) by $b\xi_n/2$, which gives:

$$\frac{b}{4}\frac{d\xi_n^2}{dt} = \frac{1}{2}\left(U - \frac{n^2\pi^2\nu}{b}\right)\xi_n^2 + \frac{\pi n}{4}\xi_n\left(\frac{1}{2}\sum_1^{n-1}\xi_h\xi_{n-h} - \sum_1^{\infty}\xi_h\xi_{n+h}\right).$$

The equation can now be considered as an analogue of Eq. (49). If we take the sum with respect to n, up to a certain value k, we can write:

$$(69) \qquad \frac{b}{4}\frac{d}{dt}\left(\sum_1^{h}\xi_n^2\right) = \frac{U}{2}\sum_1^{k}\xi_n^2 - \frac{\pi^2\nu}{2b}\sum_1^{k}n^2\xi_n^2 - W_k$$

where:

$$W_k = \frac{\pi}{4}\sum_{n=1}^{k}n\xi_n\left(\sum_{h=1}^{\infty}\xi_h\xi_{n+h} - \frac{1}{2}\sum_{h=1}^{n-1}\xi_h\xi_{n-h}\right).$$

The sums appearing here can be rearranged, as follows:

$$\sum_{n=1}^{k}\sum_{h=1}^{\infty} n\xi_n\xi_h\xi_{n+h} = \sum_{n=1}^{k}\sum_{h=k+1}^{\infty} n\xi_n\xi_{h-n}\xi_h - \sum_{n=1}^{k-1}\sum_{h=1}^{k-n} n\xi_n\xi_h\xi_{n+h} ;$$

$$\frac{1}{2}\sum_{n=1}^{k}\sum_{h=1}^{n-1} n\xi_n\xi_h\xi_{n-h} = \sum_{n=1}^{k}\sum_{h=1}^{h-1} h\,\xi_n\xi_h\xi_{n-h} =$$

$$= \sum_{h=1}^{k-1}\sum_{n=h+1}^{k} h\,\xi_n\xi_h\xi_{n-h} = \sum_{h=1}^{k-1}\sum_{n=1}^{k-h} h\,\xi_{n+h}\,\xi_h\xi_n ;$$

so that finally:

$$W_k = \frac{\pi}{4}\sum_{n=1}^{k}\sum_{h=k+1}^{\infty} n\,\xi_n\xi_{h-n}\xi_h$$

If we look at Eq. (69) and take the case represented by our first stationary solution, for which the left-hand member of the equation naturally is zero, we can observe that the term

$$\frac{1}{2}\,U\sum_{1}^{k}\xi_n^{\,2}$$

represents the energy which the turbulence derives from the mean motion U. Since with $\xi_n = 2U/n\pi$ according to the approximation (66) the series is rapidly convergent, we can say that already with a moderate value of k this sum gives all energy which is fed into the system.

We can always assume that ν is so small that for such a value of k the second sum on the right-hand side of (69), which has the value $k\pi^2\nu\,(2U/\pi)^2/2b$, is insignificant. Hence there must be a domain of values of k for which the value of W_k is practically independent of k. (This was mentioned in the paper in Adv. in Appl. Mechanics, quoted in the footnote to Section 75; see p. 180, Eq. (22d), where S_n has been used instead of W_k). This result can be compared with an argument applied in Heisenberg's theory of the turbulent spectrum.

The result can be verified by calculating the value of W_k on the basis of the approximation (66). It is more convenient to calculate $W_{k-1} - W_k$, for which, after some simple transformations of series, one finds the value $1/(k+1)^2$. This proves that W_k can be considered as a constant even for moderate values of k, to the same degree of approximation to which the sum $\sum \xi_n^2$ can be treated as a constant.

81. The question arises if a similar exponential law might be found for the amplitude function $\varphi(k)$ occurring in the Fourier integral considered in connection with the solutions representing homogeneous decaying turbulence over an infinite extent of the y-axis, not influenced by exterior forces or by coupling with a mean motion.

We observe that the introduction of the Fourier integral as given by formula (1) of Section 16 (p. 42) makes it possible to transform Eq. (1) into:

$$\frac{\partial \varphi}{\partial t} = - k^2 \nu\varphi - \frac{ik}{2}\int_{-\infty}^{+\infty} dk' \ \varphi(k') \ \varphi(k-k') \ .$$

In comparing this equation with Eq. (61) the following circumstances require attention:

(<u>a</u>) the function $\varphi(k)$ contains a phase factor which was not determined by the relation between $\varphi(k)$ and $\Gamma(k)$ and which may behave in some complicated or irregular manner;

(<u>b</u>) the absolute magnitude of $\varphi(k)$ must be proportional to $M^{1/2}$, where 2M is the length of the domain to which the Fourier integral applies;

(<u>c</u>) the dimensions of $\varphi(k)$ are (length) · (velocity), whereas the dimensions of the amplitudes ξ_n were those of a velocity.

It does not look easy to develop a formula for $\varphi(k)$ which takes account of all these facts and gives a basis for the calculation of the integral occurring in (71). The only hypothesis which presents itself is the (very tentative) assumption that for k_1 and k_2 both exceeding a certain limit K we should have:

(72) $\quad \varphi(k_1) \cdot \varphi(k_2) = 2i\nu\varphi(k_1 + k_2)$ + some irregularly fluctuating part,

where it is supposed that the "irregularly fluctuating" part does not give a contribution to the integral increasing linearly with k.

It is interesting to observe that there is a formal analogy between (72) and the relation satisfied by an exponential correlation function, as considered at the end of the Appendix to Chapter II. In both cases the only fact which matters is that an exponential relation exists, while the coefficient entering into the exponent is irrelevant. This idea would suggest that the amplitude function $\varphi(k)$ would have in it something of the nature of a correlation function, which does not look unacceptable.

82. It is of interest to look back over the results obtained with the mathematical model.

It was possible in the first place to obtain a solution representing "stimulated turbulence", produced by the action of a time dependent force at a particular spot of the field (the origin of the y-axis). The force was supposed to be a random function of the time. We had taken a system of irregularly distributed positive impulses as the simplest case, but more general cases can be studied as well. It was found that the motion produced propagated itself in the direction of the y-axis, which made it possible to consider these solutions as a rough analogue of grid-produced wind tunnel turbulence. The propagation was governed by such laws that every part of the curve where $\partial v/\partial y$ is negative, developed into a steep front which displaces itself with a velocity equal to the arithmetic mean of the velocities at the top and the bottom. Owing to the differences in velocities of the various fronts, fronts can overtake each other and then "merge" together. The consequence of this process is that, whereas originally the curve for $v(y)$ reflects most details of the time behavior of the producing force, details gradually are eliminated. The resulting turbulence thus differs in two respects from the pattern originally produced by the force: through the development of steep fronts, and through the elimination of detail. We, consequently, may say that in the long run the pattern shown by the turbulence becomes independent of the force system; only features connected with long periods in the random behavior of the force are retained over an appreciable length of time in the propagation of the turbulence.

It must be observed that this example can also be considered as illustrating in a simplified way the propagation of a system of shock waves in a tube, when waves are being produced repeatedly at the origin.

Certain results could be obtained governing the decay of the mean amplitude of the curve in its propagation.

At great distances from the origin the statistical properties of the curve for $v(y)$ change only very slowly with y, so that there is an approach to homogeneous turbulence.

A solution representing homogeneous turbulence, however, can better be obtained by starting from a case in which the v-curve over the whole y-axis is produced at a single instant by a system of impulsive forces acting simultaneously in a system of points, arbitrarily distributed, after which the system is left to itself. Initially the v-curve will picture the distribution of the original impulsive forces, but again there is the tendency to produce steep fronts wherever $\partial v/\partial y$ happens to be negative, and following that more and more details will be gradually eliminated through the merging of fronts.

This case was appropriate for an investigation of spatial Eulerian correlations of the type $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$. An equation could be formed, which can be considered as a very simplified analogue of von Karman and Howarth's equation applied in the theory of homogeneous isotropic turbulence. It is also possible to obtain a Fourier integral representing the solutions, and the amplitude function appearing in the integral is related to the spectral function in the same way as in other correlation theories. A number of properties considered in the theory of homogeneous hydrodynamic turbulence can be illustrated by means of analogous properties of the solutions for the model.

The problem was considered whether in this case there is a tendency to assume a pattern developing according to a law of similarity. Certain aspects were brought to light which seem to speak against this. It may be that these aspects are peculiar to the model and that in hydrodynamic turbulence, with its three-dimensional field, matters will be different.

The last case considered concerned turbulence coupled with a quantity representing something like the mean flow of the fluid in pipe flow. To obtain a satisfactory example it was necessary to complete Eq.(1) with a coupling term, as was done in Eq. (51) The field had been limited to the domain $0 \leq y \leq b$ and v had been subjected to the boundary condition

$v = 0$ at both "walls". The turbulence could be analyzed into a Fourier series so that it was easy to speak of separate components of turbulence. The equation governing these components was studied, which revealed that depending on the Reybolds number, a "laminar" solution and a "turbulent" solution is possible. Expressions for the components could be obtained by deriving stationary turbulent solutions directly from the partial differential equation and calculating the Fourier coefficients for these solutions. The formula giving the values of the amplitudes ξ_n shows that at the head of the spectrum they decrease inversely proportional to the index n, in such a way that the dissipation of energy has the same value for each component. At the tail end of the spectrum an exponential law is obtained which makes the sum of the dissipation terms convergent.

The circumstance that the turbulent solutions for this system were found to assume a time-independent form, is exceptional and is due to the simple form which was given to Eqs. (1) and (51). It is possible to construct other systems, slightly more complicated, so that the turbulent motion has two components, v and w, where the turbulent solution must be time-dependent. However, although it can be shown that the equations for such a system have many properties in common with the simpler equations considered here, it has not been possible to discuss them so fully.

Chapter VIII

Homogeneous Isotropic Turbulence

The preceding chapter gave an extensive investigation of some problems connected with a nonlinear equation characteristic for a dissipative system. The simplest type of equation had been chosen, referring to a single variable which was a function of one coordinate and of the time. It is impossible at present to develop a similar treatment for the hydrodynamic equations. However, a great amount of research has been carried out on the correlation problems connected with these equations and on the theory of their spectra. This research has had its greatest success in the domain of homogeneous isotropic turbulence where considerations of symmetry make it possible to express certain sets of correlation functions with the aid of a single function. Equations can then be obtained for the basic functions expressing relations pertaining to energy decay and to the transmission of energy through the spectrum. A short account of this theory will be given in this last chapter. The principal mathematical relations will be deduced in a somewhat condensed form, and the reader is referred to original publications for a number of proofs. The main object is to obtain relations for which the model system gave a simplified form. This will show for which purpose the model system had been introduced, and a comparison with the treatment developed for the model system may give some idea of the problems still concealed behind the equations for hydrodynamic turbulence.

All mean values to be introduced in the following pages will be space mean value of the Eulerian type. They will be functions of the time if the field is decaying. Although the dependence on the time will not be indicated explicitly in the expressions, one of the objects is to obtain equations determining the time derivatives.

83. Ordinary (or double) Velocity Correlations. The first quantity to be considered is the correlation product:

$$Q_{ij} = \overline{u_i u_j'} \tag{1}$$

formed out of two components of the velocity, u_i observed at a point P with coordinates x_1, x_2, x_3 and u_j' observed at a point P' with coordinates

$x_1 + r_1$, $x_2 + r_2$, $x_3 + r_3$. Since the field is homogeneous, the mean value will depend only on the relative position of P' with respect to P, as determined by the r_i. The absolute value of the distance follows from $r^2 = r_i^2$. To simplify notation, summation signs are omitted. It is understood that summation is carried out over any repeated index.

In principle there are nine correlation functions, each depending on the three relative coordinates r_i in its own way. When the field is isotropic, meaning that its statistical properties are not dependent on a direction fixed in space, these nine functions can be expressed by means of two functions $Q_I(r)$, $Q_{II}(r)$, as follows:

$$Q_{ij} = Q_I \, r_i r_j + Q_{II} \, \delta_{ij} \, , \tag{2}$$

where δ_{ij} denotes a tensor which is equal to unity when $i = j$ and equal to zero when $i \neq j$.

The condition of isotropy also has the consequence that

$$Q_{ij} = Q_{ji} \; .$$

This is implied in (2) and can be seen by observing that the correlation product must not change when we interchange P and P' and simultaneously change the signs of the coordinates, which changes the signs of the r_i and also that of the u_i and u_j'.

If we take $r_2 = r_3 = 0$, so that PP' is parallel to the x_1-axis and $r = r_1$, we have:

$$\overline{u_1 \, u_1'} = Q_I \, r^2 + Q_{II}$$

$$\overline{u_2 \, u_2'} = Q_{II}$$

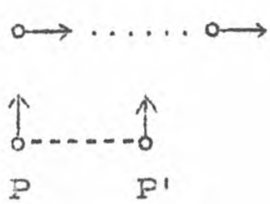
$$\overline{u_1 \, u_2'} = 0 \; ; \quad \overline{u_2 \, u_3'} = 0 \; .$$

For $r = 0$ (P and P' coinciding) we have:

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = Q_{II}(0) \, ,$$

for which one also writes $\overline{u^2}$, so that $\overline{u^2}$ is the mean value of the square of a component of the velocity taken in an arbitrary given direction. The mean value of the square of the absolute velocity is $\overline{3u^2} = 3\,Q_{II}(0)$.

It is customary to introduce two normalized correlation functions, $f(r)$ and $g(r)$, defined by means of the following diagrams:



$\overline{u^2} \cdot f(r)$

$\overline{u^2} \cdot g(r)$

If we return to the case $r_2 = r_3 = 0$, $r = r_1$, we find:

$$(3) \qquad \begin{cases} Q_{11} = \overline{u_1\, u_1'} = \overline{u^2} \cdot f(r) \\ Q_{22} = \overline{u_2\, u_2'} = \overline{u^2} \cdot g(r) \end{cases}$$

84. In the case of an incompressible fluid the equation of continuity enables us to obtain a relation between Q_I and Q_{II}. We have:

$$\partial u_j/\partial x_j = 0\,,$$

and consequently:

$$u_i \cdot \partial u_j'/\partial r_j = 0\,.$$

This immediately leads to:

$$\partial\, Q_{ij}/\partial r_j = 0.$$

When this is worked out on the basis of (2) we find:

$$r\,Q_1' + 4\,Q_I + Q_{II}'/r = 0$$

(accents here denoting derivatives with respect to r). Hence we can express Q_{II} in Q_I or both Q_I and Q_{II} in terms of an auxiliary function $Q(r)$. The form ordinarily chosen is:

$$Q_I = -Q'/2r\,; \quad Q_{II} = \frac{1}{2}\,Q'\,r + Q, \quad \text{so that } Q_I\,r^2 + Q_{II} = Q.$$

It is then possible to write (2) in the form:

$$(4) \qquad Q_{ij} = -(Q'/2r) \cdot r_i r_j + (Q + \frac{1}{2} Q' r)\, \delta_{ij}\,.$$

With $r_2 = r_3 = 0$, $r = r_1$, we now have:

$$Q_{11} = \overline{u^2} \cdot f(r) = Q$$

$$Q_{22} = \overline{u^2} \cdot g(r) = Q + \frac{1}{2} Q' r\,.$$

From the Q_{ij} we derive the function R:

$$R = Q_{ii} = Q_{11} + Q_{22} + Q_{33} = u_1 u_1' + u_2 u_2' + u_3 u_3' = 3\,Q + Q' r\,.$$

(5)

Whereas Q_{11}, Q_{22}, Q_{33} are dependent on the orientation of the vector r giving the distance between P and P', the function R is dependent only on the absolute value of r.

For $r = 0$ we have:

$$R(0) = 3\,Q(0) = 3\,\overline{u^2}\,.$$

It is also easily found that:

$$\int_0^r R\, r^2\, dr = Q\, r^3\,.$$

If we may assume that for indefinitely increasing r the function $Q(r)$ goes to zero sufficiently rapidly to make $\lim (Q\, r^3) = 0$, we find:

$$\int_0^\infty R\, r^2\, dr = 0\,.$$

This shows that somewhere R must have negative values and the same must be true of the sum $Q_{11} + Q_{22} + Q_{33}$.

85. Pressure-Velocity Correlations. We take the value p of the pressure at the point P and a velocity component u_j' at P', and consider the mean value of the product of these two quantities. It is readily seen

that in an isotropic field one can expect a correlation different from zero only between the pressure at P and the velocity component u_r' at P' in the direction of PP' (or in the inverse direction). Now if this correlation would be different from zero, it must have the same value for all directions of PP', but this would mean that there would be an average radial flow of fluid, either away from P if the correlation were positive or towards P if the correlation were negative. This is impossible when there is no source or sink at P. Consequently we must conclude that in the absence of sources and sinks the pressure-velocity correlation is always zero in isotropic turbulence. Hence:

$$(7) \qquad \overline{p\, u_j'} = 0 \quad \text{for all } j.$$

86. Triple Velocity Correlations. Equations for the behavior of the double correlations as functions of the time can be deduced from the hydrodynamic equations:

$$(8) \qquad \frac{\partial u_i}{\partial t} + u_h \frac{\partial u_i}{\partial x_h} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\, \Delta u_i \,.$$

The equation referring to u_i at P is multiplied by u_j' (at P'), a similar equation for u_j' at P' is multiplied by u_i, the results are added and the mean value is taken. This gives:

$$(9) \qquad \frac{\partial}{\partial t}\,\overline{u_i u_j'} + \left(\overline{u_h \frac{\partial u_i}{\partial x_h}\, u_j'} + \overline{u_i u_h' \frac{\partial u_j}{\partial x_h'}}\right) = 2\,\nu\,\Delta\,\left(\overline{u_i u_j'}\right) \,.$$

Here a new type of mean value appears, referring to products of the third degree. This leads to the investigation of triple correlation products:

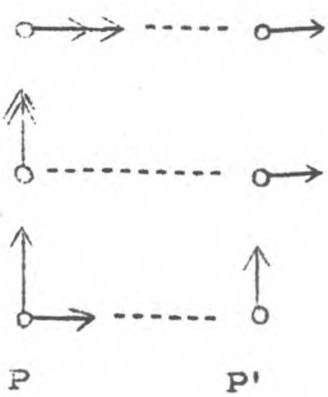
$$(10) \qquad T_{ihj} = \overline{u_i\, u_h\, u_j'} \,,$$

where u_i and u_h are taken at P and u_j' at P'.

In the case of isotropic turbulence the tensor T_{ihj} can be expressed with the aid of three functions of r, as follows:

$$(11) \qquad T_{ihj} = T_{hij} = T_I \cdot r_i r_h r_j + T_{II}(r_i\,\delta_{hj} + r_h\,\delta_{ij}) + T_{III}\, r_j\,\delta_{ih} \,.$$

The functions T_I, T_{II}, T_{III} are related to three basic types of triple correlation functions defined by means of the following diagrams. (Each arrow indicates a velocity component; a double arrow indicates a component squared).



$\overline{(u^2)}^{3/2} \cdot k(r)$

$\overline{(u^2)}^{3/2} \cdot h(r)$

$\overline{(u^2)}^{3/2} \cdot q(r)$

If again we take $r_2 = r_3 = 0$, $r = r_1$, we have:

$$(12) \quad \begin{cases} T_{111} = T_I\, r^3 + 2\, T_{II}\, r + T_{III}\, r = \overline{(u^2)}^{3/2} \cdot k(r) \\ T_{221} = T_{III}\, r = \overline{(u^2)}^{3/2} \cdot h(r) \\ T_{122} = T_{II}\, r = \overline{(u^2)}^{3/2} \cdot g(r) . \end{cases}$$

From T_{ihj} we can form:

$$T_{iij} = \overline{(u_1^{\,2} + u_2^{\,2} + u_3^{\,2})\, u_j'} \; .$$

Since the first factor is a scalar, independent of direction, this mean value must be zero on the same grounds as made $\overline{p\, u_j'}$ zero. This gives a relation between T_I, T_{II}, and T_{III}. Two more relations are obtained from the equation of continuity, which gives:

$$\partial\, T_{ihj} / \partial\, r_j = 0 \; .$$

One relation appears to be a consequence of the two others and all three functions can be expressed by means of a single one. The following form can be chosen:

$$(13) \quad \begin{cases} T_I = 2\,\theta + \theta'\, r \\ T_{II} = -\frac{5}{2}\,\theta\, r^2 - \frac{1}{2}\,\theta'\, r^3 \\ T_{III} = \theta\, r^2 \, , \end{cases}$$

where θ is a function of r. For certain purposes it is useful to introduce a function $T = 10\, T_{III} + 2\, T_{III}'\, r$; then:

$$(13a) \qquad T = 14\, \theta\, r^2 + 2\, \theta'\, r^3 .$$

The expressions for the functions $k(r)$, $h(r)$, $q(r)$ in terms of θ are as follows:

$$(13b) \qquad \begin{cases} \overline{(u^2)}^{3/2} \cdot k(r) = -2\, \theta\, r^3 \\ \overline{(u^2)}^{3/2} \cdot h(r) = +\, \theta\, r^3 \\ \overline{(u^2)}^{3/2} \cdot q(r) = -\frac{5}{2}\, \theta\, r^3 - \frac{1}{2}\, \theta'\, r^4 . \end{cases}$$

Triple correlation products in which two components refer to the point P' and only one component to P, can be obtained from those considered thus far by interchanging the points P and P'. In isotropic turbulence the value of any triple correlation product does not change by this process, provided we simultaneously change the signs of the r_1 and of the velocity components. It follows that one has the relation:

$$(14) \qquad \overline{u_i\, u_h\, u_j'} = -\, \overline{u_j\, u_i'\, u_h'}$$

87. A new quantity is now derived from the T_{ihj} by means of the formula:

$$(15) \qquad T_{ij} = T_{ji} = -\frac{\partial}{\partial r_h}(T_{ihj} + T_{jhi}) .$$

Since we have:

$$\frac{\partial}{\partial r_h} T_{ihj} = \frac{\partial}{\partial r_h'} \overline{(u_i u_h u_j')} = -\frac{\partial}{\partial r_h} \overline{(u_i u_h u_j')} = -\, \overline{u_h \frac{\partial u_i}{\partial r_h}\, u_j'}$$

$$\frac{\partial}{\partial r_h} T_{jhi} = \frac{\partial}{\partial r_h'} \overline{(u_j u_h u_i')} = -\frac{\partial}{\partial r_h'} \overline{(u_i u_j' u_h')} = -\, \overline{u_i u_h' \frac{\partial u_j'}{\partial r_h'}} ,$$

it follows that:

(16) $$T_{ij} = u_h \frac{\partial u_i}{\partial x_h} u_j' + u_i u_h' \frac{\partial u_j'}{\partial x_h} .$$

which is just the term occurring in Eq. (9). This equation consequently can be written:

(17) $$\frac{\partial Q_{ij}}{\partial t} + T_{ij} = 2 \nu \Delta Q_{ij} .$$

It is convenient to make use of the following reductions. By substituting the expression (11) into (15) and working out the differentiations, the result is obtained:

$$T_{ij} = -\left[10\, T_I + 2\, T_I' r + \frac{2 T_{II}'}{r} + \frac{2 T_{III}'}{r} \right] r_i r_j - \left[8\, T_{II} + 2\, T_{II}' r + 2\, T_{III} \right] \delta_{ij}$$

and by making use of (13)

$$T_{ij} = -\left[14\, \theta + 10\, \theta' r + \theta'' r^2 \right] r_i r_j + \left[28\, \theta r^2 + 12\, \theta' r^3 + \theta'' r^4 \right] \delta_{ij} .$$

Having regard to (13a) this can also be written:

(18) $$T_{ij} = -\, (T'/2r)\, r_i r_j + (T + \frac{1}{2} T' r)\, \delta_{ij} .$$

From formula (4) one obtains:

$$\Delta Q_{ij} = \left[\frac{2Q'}{r^3} - \frac{2Q''}{r^2} - \frac{Q'''}{2r} \right] r_i r_j + \left[\frac{2Q'}{r} + 3\, Q'' + \frac{1}{2} Q''' r \right] \delta_{ij} .$$

If we write:

$$D = \frac{4Q'}{r} + Q'' ,$$

this reduces to:

(19) $$\Delta Q_{ij} = -\, (D'/2r)\, r_i r_j + (D + \frac{1}{2} D' r)\, \delta_{ij} .$$

88. The expressions (4), (18) and (19) all have the same structure. It follows that equation (17) will be satisfied if:

(20) $$\frac{\partial Q}{\partial t} + T = 2\, \nu D$$

Another equation can be deduced from (17) by taking $i = j$, which requires summation with respect to i. As follows from (5), the first term of (17) then becomes: $\partial R/\partial t$. We further write:

$$(21) \quad S = T_{ii} = T_{11} + T_{22} + T_{33} = 3T + T'r = 70\,\theta r^2 + 26\,\theta' r^3 + 2\,\theta'' r^4 .$$

Since:

$$\Delta Q_{ii} = \Delta(Q_{11} + Q_{22} + Q_{33}) = \Delta R ,$$

we obtain:

$$(22) \qquad \frac{\partial R}{\partial t} + S = 2\,\nu\,\Delta R .$$

Equation (20) can still be transformed by returning to the normalized correlation functions $f(r)$ and $h(r)$. We have:

$$Q = \overline{u^2} \cdot f(r)\,; \quad T = 2\,(\overline{u^2})^{3/2} \cdot (4h/r + h')\,; \quad D = \overline{u^2} \cdot (4f'/r + f'') ,$$

and find:

$$(23) \quad \frac{\partial}{\partial t}(\overline{u^2}\,f) + 2\,(\overline{u^2})^{3/2}\left(\frac{4f}{r} + h'\right) = 2\;\overline{u^2}\left(\frac{4f'}{r} + f''\right) .$$

This is the fundamental equation for the propagation of correlation, as given by von Karman and Howarth.

89. Additional Remarks. The function $Q(r)$ is an even function of r, which starts with a term independent of r. The same applies to the functions Q_i, Q_{II}, R, and also to $f(r)$ and $g(r)$.

The function $T(r)$ likewise is an even function of r, which, however, starts with a term in r^2 (see Eq. 13a). The function $\theta(r)$ is even and starts with a term independent of r. It follows from (13) that T_I starts with a term independent of r, whereas T_{II} and T_{III} start with terms in r^2. The functions $k(r)$, $h(r)$, and $q(r)$ are odd functions of r, which start with terms in r^3.

It will be seen from (18) that $T_{ij}(0) = 0$.

If we multiply Eq. (21) through with r^2, we obtain:

$$\frac{\partial(Rr^2)}{\partial t} + \frac{\partial}{\partial r}(Tr^3) = 2\,\nu\frac{\partial}{\partial r}\left(r^2\,\frac{\partial R}{\partial r}\right)$$

if regard is given to the expression (21) for S and to the formula for ΔR in spherical polar coordinates. If we assume that Tr^3 and $r^2(\partial R/\partial r)$ vanish for $r = \infty$, integration gives:

$$\frac{d}{dt}\int_0^\infty R\,r^2\,dr = 0.$$

This is in accordance with the result previously obtained, according to which

$$\int_0^\infty R\,r^2\,dr = 0\,.$$

If we multiply with r^4, we obtain

$$\frac{\partial(R\,r^4)}{\partial t} + \frac{\partial}{\partial r}(10\,\theta\,r^7 + 2\,\theta' r^8) = 2\,\nu\,r^2\,\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right).$$

Integration now gives:

$$\frac{d}{dt}\int_0^\infty R\,r^4\,dr = 12\,\nu\int_0^\infty R\,r^2\,dr = 0\,.$$

Hence

$$\int_0^\infty R\,r^4\,dr$$

is independent of the time. If we express R by means of Q according to (5), we also find that

$$J = \int_0^\infty Q\,r^4\,dr \tag{24}$$

is independent of the time. This integral is called Loitsiansky's invariant for homogeneous isotropic turbulence. The result also can be deduced from Eq. (23) by multiplying this equation by r^4 and integrating it, since

$\overline{u^2} \cdot f(r) = \Omega.$

90. Our main interest now is in the spectral analysis connected with the functions introduced in the preceding pages.

Since the field is three-dimensional, a triple Fourier integral is needed. In extension of formula (1) of Section 16, we use the following representation for the velocity component $u_i(x_1, x_2, x_3)$, which shall be valid within a cubical region of space with sides equal to $2M$:

$$(25) \qquad u_1 = \iiint_{-\infty}^{+\infty} dk_1\, dk_2\, dk_3\ \varphi_i(k_1, k_2, k_3)\, e^{i(k_1x_1 + k_2x_2 + k_3x_3)} .$$

It is expected that no confusion will arise through the use of $i = \sqrt{-1}$ along with i as an index. The amplitude function must satisfy the condition

$$\varphi_i(k_1, k_2, k_3) = \varphi_i{}^*(-k_1, -k_2, -k_3) ,$$

where the asterisk denotes the complex conjugate function. The amplitude function can depend on the time, but it is not necessary to state this explicitly. There is a function φ_i corresponding to each of the three components u_i. The equation of continuity leads to the following relation between these functions:

$$(26) \qquad k_1 \cdot \varphi_1 + k_2 \cdot \varphi_2 + k_3 \cdot \varphi_3 = 0 .$$

In order to reduce writing, we introduce the vectors $\underline{x}$ and $\underline{k}$, replace $k_1x_1 + k_2x_2 + k_3x_3$ by the scalar product $\underline{k} \cdot \underline{x}$, and abbreviate $dk_1\, dk_2\, dk_3$ by $d\underline{k}$, writing a single integral sign only. Further, in the same way as before, summation signs will be omitted. Summation has to be carried out every time an index is repeated. Equations (25) and (26) then become:

$$(25a) \qquad u_i = \int d\underline{k}\ \varphi_i(\underline{k})\, e^{i\,\underline{k}\cdot\underline{x}}$$

$$(26a) \qquad k_i\, \varphi_i = 0 .$$

The inverse of (25a) is the formula:

$$(27) \qquad \varphi_i(\underline{k}) = \frac{1}{8\pi^3} \int_{-M}^{+M} u_i(\underline{x})\, e^{-i\,\underline{k}\cdot\underline{x}}\, d\underline{x} .$$

91. To obtain Fourier integrals for the correlation functions, we follow a similar procedure as was applied in Section 17. Since the coordinates of P determine the vector $\underline{x}$ and those of P' the vector $\underline{x} + \underline{r}$, we have:

$$u_i u_j' = \int_{-\infty}^{+\infty} d\underline{k}' \int_{-\infty}^{+\infty} d\underline{k}'' \ \varphi_i(\underline{k}')\ \varphi_j(\underline{k}'')\ e^{i(\underline{k}'+\underline{k}'' \cdot \underline{x} + i\, \underline{k}'' \cdot \underline{r}} .$$

To obtain the mean value we integrate with respect to x_1, x_2, x_3 from $-M$ to $-M_1$, where $M_1 > M$ and divide by $8M^3$. This gives:

$$\overline{u_i u_j'} = \frac{1}{M^3} \int d\underline{k}' \int d\underline{k}'' \varphi_i(\underline{k}')\ \varphi_j(\underline{k}'')\ e^{i\underline{k}'' \cdot \underline{r}} \frac{\sin(k_1'+k_1'')M_1 \cdot \sin(k_2'+k_2'')M_2 \cdot \sin(k_3'+k_3'')M_3}{(k_1'+k_1'')(k_2'+k_2'')(k_3'+k_3'')}$$

which for $M_1 \rightarrow \infty$ transforms into (writing $\underline{k}$ for $\underline{k}'$):

$$\frac{\pi^3}{M^3} \int d\underline{k}\ \ \varphi_i(\underline{k})\ \ \varphi_j(-\underline{k})\ e^{-i\underline{k} \cdot \underline{r}} .$$

Hence if we introduce:

$$\Gamma_{ij}(\underline{k}) = \frac{\pi^3}{M^3}\ \varphi_i(\underline{k})\ \ \varphi_j(-\underline{k}) ,$$

we have:

$$(28) \qquad Q_{ij} = \overline{u_i u_j'} = \int d\underline{k}\ \ \Gamma_{ij}(k)\ e^{-i\underline{k} \cdot \underline{r}}$$

In isotropic turbulence the Γ_{ij} form a tensor in the $\underline{k}$-space of similar nature as the Q_{ij}, so that, in analogy with (2):

$$\Gamma_{ij} = A(k) \cdot k_i k_j + B(k)\ \delta_{ij} ,$$

where $A(k)$ and $B(k)$ only depend on the absolute value of $k =$

$$= \sqrt{k_1^2 + k_2^2 + k_3^2} .$$

Since the equation of continuity gives:

$$k_j \cdot \Gamma_{ij} = 0 ,$$

it is possible to express the function $A(k)$ by means of $B(k)$ and we can write:

$$\Gamma_{ij} = B(k)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{29}$$

When we introduce:

$$\Gamma = \Gamma_{ii} = 2B$$

we find:

$$R(r) = \int d\underline{k}\ \Gamma(\underline{k})\, e^{-i\,\underline{k}\cdot\underline{r}} .$$

Since $\Gamma(k)$, like $B(k)$, depends on the absolute value of k only, this integral can be transformed by introducing polar coordinates into the $\underline{k}$-space, using the direction of the vector $\underline{r}$ as polar axis. If ϑ is the angle between the vector $\underline{k}$ and the vector $\underline{r}$, we have $\underline{k}\cdot\underline{r} = k\,r\cos\vartheta$; for $\underline{dk}$ we substitute $2\pi\, k^2 \sin\vartheta\, dk$, and obtain:

$$R(r) = 4\pi \int_0^\infty \Gamma(k)\,\frac{k \sin kr}{r}\, dk . \tag{30}$$

It is convenient to write:

$$4\pi k^2\, \Gamma(k) = F(k) ,$$

so that:

$$R(r) = \int_0^\infty F(k)\,\frac{\sin kr}{kr}\, dk . \tag{30a}$$

For $r = 0$ this reduces to:

$$R(0) = \int_0^\infty F(k)\, dk . \tag{30b}$$

Since $R(0)$ is equal to the mean square value of the absolute velocity and thus apart from a factor $\frac{1}{2}$ measures the mean kinetic energy per unit volume, this formula gives the spectral resolution of the kinetic energy, in the sense that $F(k)\, dk$ represents the contribution to the kinetic energy, derived from harmonic components with an absolute spatial frequency k (wavelength $2\pi/k$).

From the theory of Fourier integrals it follows that the inverse of (30a) is:

$$\text{(30c)} \qquad F(k) = \frac{2k}{\pi} \int_0^\infty r \sin kr\, R\, dr .$$

To obtain the spectral resolution of the kinetic energy for a single component, we observe that according to (29):

$$\overline{u_1^2} = C_{11} = \int d\underline{k}\, \Gamma_{11}(\underline{k}) = \int d\underline{k}\, B(k)\left(1 - \frac{k_1^2}{k^2}\right) = \int_{-\infty}^{+\infty} dk_1 \int_0^\infty 2\pi k''\, B(k)\left(1 - \frac{k_1^2}{k^2}\right) dk'',$$

where $k'' = \sqrt{k_2^2 + k_3^2}$. Since $k''^2 = k^2 - k_1^2$, we have $k''\, dk'' = k\, dk$ for constant k_1. Hence if we write:

$$\text{(31)} \qquad F_1(k_1) = \int_{k_1}^\infty \frac{F(k)}{2k}\left(1 - \frac{k_1^2}{k^2}\right) dk ,$$

we find:

$$\text{(31a)} \qquad \overline{u_1^2} = \int_0^\infty F_1(k_1)\, dk_1 .$$

This is often denoted as the formula for the one-dimensional or Taylor-spectrum.

We finally observe:

$$\text{(32)} \qquad \Delta R = - \int_0^\infty k^2 F(k) \frac{\sin kr}{kr}\, dk$$

92. A calculation similar to the one applied for obtaining Γ_{ij} gives:

$$T_{ihj} = \overline{u_i u_h u_j'} = \frac{\pi^3}{M^3} \int d\underline{k}' \int d\underline{k}''\, \varphi_i(\underline{k}')\, \varphi_h(\underline{k}'')\, \varphi_j(-\underline{k}' - \underline{k}'')\, e^{-i(\underline{k}' + \underline{k}'')\cdot \underline{r}} .$$

We write $\underline{k}' + \underline{k}'' = \underline{k}$; hence $\underline{k}'' = \underline{k} - \underline{k}'$; the integration with respect to $d\underline{k}'$ and $d\underline{k}''$ can then be replaced by one with respect to $d\underline{k}$ and $d\underline{k}'$.
In this way we obtain:

$$\text{(33)} \qquad T_{ihj} = \int d\underline{k}\, \Psi_{ihj}(\underline{k})\, e^{-i\,\underline{k}\cdot\underline{r}} ;$$

where:

(33a) $$\Psi_{ihj}(\underline{k}) = \frac{\pi^3}{M^3} \int d\underline{k}'\, \varphi_i(\underline{k}')\, \varphi_h(\underline{k}-\underline{k}')\, \varphi_j(-\underline{k}) .$$

We shall also write:

(34) $$T_{ij} = \int d\underline{k}\ \Psi_{ij}(\underline{k})\, e^{-i\,\underline{k}\cdot\underline{r}} ;$$

then:

$$\Psi_{ij} = \Psi_{ji} = i\,k_h(\Psi_{ihj} + \Psi_{jhi}) = \frac{\pi^3 i}{M} \int d\underline{k}' \left\{ k_1\, \varphi_1(\underline{k}-\underline{k}') + k_2\, \varphi_2(\underline{k}-\underline{k}') + k_3\, \varphi_3(\underline{k}-\underline{k}') \right\} \cdot \left\{ \varphi_i(\underline{k}')\, \varphi_j(-\underline{k}) + \varphi_j(\underline{k}')\, \varphi_i(-\underline{k}) \right\} .$$

In order to be able to express $S = T_{ii}$ by means of a Fourier integral, we must form:

(35) $$\Psi = \Psi_{ii} = \frac{2\pi^3 i}{M^3} \int d\underline{k}' \left\{ k_1\, \varphi_1(\underline{k}-\underline{k}') + k_2\, \varphi_2(\underline{k}-\underline{k}') + k_3\, \varphi_3(\underline{k}-\underline{k}') \right\} \cdot \left\{ \varphi_1(\underline{k}')\, \varphi_1(-\underline{k}) + \varphi_2(\underline{k}')\, \varphi_2(-\underline{k}) + \varphi_3(\underline{k}')\, \varphi_3(-\underline{k}) \right\} .$$

This is a function of the absolute value of k only. Hence we obtain:

(36) $$S = T_{ii} = \int d\underline{k}\ \Psi\, e^{-i\,\underline{k}\,\underline{r}} = \int_0^\infty w(k)\, \frac{\sin k r}{k r}\, dk ,$$

with

(36a) $$4\pi k^2\, \Psi(k) = w(k) .$$

The inverse formula of (36) is:

(36b) $$w(k) = \frac{2k}{\pi} \int_0^\infty r \sin k r\, S\, dr .$$

For $r = 0$ we have $S(0) = 0$; hence it follows from (36) that

(36c) $$\int_0^\infty w(k)\, dk = 0.$$

93. We can now translate Eq. (22) into a relation between the quantities $F(k)$ and $w(k)$, as follows:

(37) $$\frac{\partial F(k)}{\partial t} = - w(k) - 2 \nu k^2 F(k).$$

This equation is an analogue of Eq. (49) given in Section 71 for the simple mathematical model, the function $- w(k)$ playing the part of $k\Psi$ occurring in (49). It is often given in an integrated form, with limits 0 and k. We shall write:

$$\int_0^k w(k)\, dk = W(k) ;$$

then:

(38) $$\frac{\partial}{\partial t} \int_0^k F(k)\, dk = - W(k) - 2 \nu \int_0^k k^2 F(k)\, dk.$$

This equation is an analogue of Eq. (50) of Section 71 (p. 155). It is also an analogue of Eq. (69) of Section 80 (p. 169), provided the term with the factor U/2 is omitted. The left-hand side of Eq. (38) gives the change of energy (apart from the factor 1/2) associated with harmonic components of wave numbers not exceeding k (wavelength $\geq 2\pi/k$). The second term on the right-hand side gives the loss of energy from this part of the spectrum through viscous decay; the first term gives the loss from this part of the spectrum through interaction with the rest of the spectrum. It is evident from its definition that $W(0) = 0$, but we have also proved that W vanishes at infinity (see 36c).

Equation (38) forms the starting point for certain deductions by Heisenberg and by von Karman and others, who all make use of the hypothesis that W(k) may be represented by an expression consisting of the product of two independent integrals, as follows:

$$\text{const.} \left(\int_k^\infty F^{\alpha} k^{\beta}\, dk \right) \cdot \left(\int_0^k F^{\alpha'} k^{\beta'}\, dk \right) .$$

We shall not enter into these considerations since they have been treated very fully in the literature. The corresponding expression for the model system, that is, the integral -

$- \int_0^k k\, \Psi\, dk$ occurring in Eq. (50) of page 155 can be brought into the form of a double integral of the following type:

$$-\frac{4\pi i}{M}\int_0^k dk_1\, k_1\, \varphi(k_1)\int_k^\infty dk_2\, \varphi(-k_2)\,\varphi(k_2-k_1)\,.$$

It will be seen that this has a character differing from that assumed by Heisenberg.

If follows from (26a) that w(k) vanishes whenever the vectors $\underline{k}'$ and $\underline{k}$ have the same direction, which entails that also $\underline{k} - \underline{k}'$ will have that same direction. (Remark made by Dr. Ch. M. Tchen in Washington.) The function w(k) vanishes likewise when the scalar product of the complex quantities $\varphi(\underline{k}')$ and $\varphi(-\underline{k})$ is zero.

It can be attempted to find a condition, analogous to Eq. (72) of Section 81 (p. 171), which might ensure that the terms of the highest degree in k will cancel in (37). Owing to the three-dimensional structure of the formulas, the result is more complicated. A possible formulation seems to be:

$$\left.\begin{array}{l}\text{mean value for all}\\ \text{directions of the}\\ \text{vector } k'\text{, the abso-}\\ \text{lute value of } k' \text{ re-}\\ \text{maining constant}\end{array}\right\} \text{ of } \left[\frac{k_j}{k}\,\varphi_j(\underline{k}-\underline{k}')\right]\cdot\varphi_i(\underline{k}')$$

$$\text{should be equal to } \frac{3i}{4\pi}\,\frac{\nu}{k^2}\,\varphi_i(\underline{k})\,,$$

for a given vector $\underline{k}$, of arbitrary direction and magnitude, provided the absolute values $|\underline{k}'|$ and $|\underline{k}-\underline{k}'|$ both exceed a fixed number K.

The appearance of k^2 in the denominator of the right-hand side is connected with the circumstance that φ has the dimensions

$$(\text{velocity})\cdot(\text{length})^3.$$

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